Unsteady convection problems

Problem formulation

$$\begin{cases} u_t + a u_x = 0 & x \in (0,1), t \in (0, 0.6] \\ u(x,0) = u_0(x) & x \in (0,1) \\ u(0,t) = 0 & t \in (0, 0.6] \end{cases}$$

$$u_0(x) = \begin{cases} 1 & \text{if } x \leq 0.2 \\ 0 & \text{otherwise} \end{cases}$$

$$a = 1, \Delta x = 2 \cdot 10^{-2}, \Delta t = 1.5 \cdot 10^{-2}$$

Task 1

Compute the Courant number. Solve the problem using the Crank-Nicolson scheme in time and linear finite element for the Galerkin scheme in space. Is the solution accurate?

The Courant number is defined as

$$C = \frac{a \Delta t}{\Delta x}.$$

For the given problem the Courant number is

$$C = 0.75.$$

The consistent mass matrix expressed in terms of the shape function on an element level can be written as

$$M^e_{ab} = \int_{\Omega^e} \begin{pmatrix} N_1 N_1 & N_1 N_2 \\ N_2 N_1 & N_2 N_2 \end{pmatrix} dx$$

Additionally to this standard finite element approach, the provided MATLAB code was extended by a lumped mass matrix for the Crank-Nicolson scheme as well as the Lax-Wendroff scheme which will be introduced in Task 2. The lumped or diagonal mass matrix is commonly used in the finite difference approach but it can also be applied to the finite element method and is written as [1]

$$[M^L]^e_{ab} = \int_{\Omega^e} \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} dx$$

Both approaches for the Crank-Nicolson scheme are shown in Figure 1 and will be discussed in the following paragraph.

The solution for solving the presented problem using the Crank-Nicolson scheme in time and the Galerkin scheme for a linear finite element in space at time step $t = 0.6$ is shown in Figure 1 (left). It can be observed that the accuracy of the result is diminished by multiple oscillations induced by the Galerkin method which are distributed over the domain before the discontinuity[1].

The solution for the Crank-Nicolson scheme with a lumped mass matrix is displayed in Figure 1 (right). In comparison two the solution in Figure 1 (left) increased oscillations and a larger offset close to the sharp gradient can be observed. Overall, the accuracy regarding the exact solution has decreased even more.

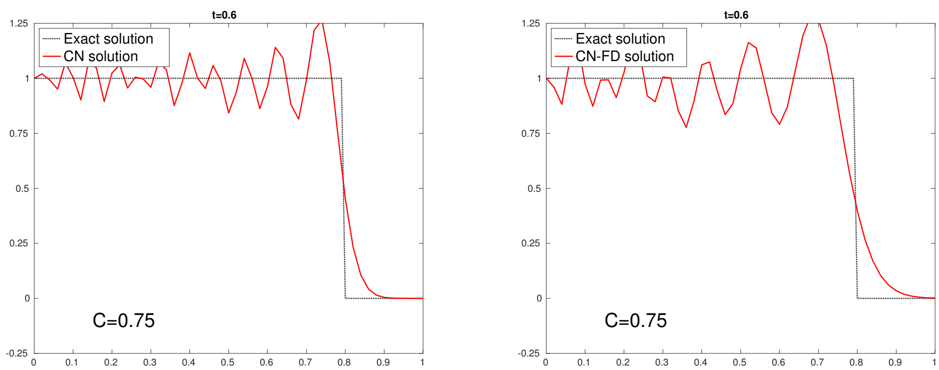

Figure 1: Comparison of the Crank-Nicolson scheme with (right) and without (left) a lumped mass matrix

Task 2

Solve the problem using the second-order Lax-Wendroff method. Can we expect the solution to be accurate? If not, what changes are necessary? Comment the results.

Figure 2 (left) shows the solution of solving the unsteady convection problem using the Lax-Wendroff scheme in time and the Galerkin scheme for a linear finite element in space at time step $t = 0.6$. For the given Courant number the solution is completely unstable resulting in heavy oscillation over the whole domain. In order to overcome this drawback a lumped mass matrix as described in Task 1 has been implemented. The result is displayed in Figure 2 (right) showing an accurate solution with only small oscillations before and a minor offset around the discontinuity.

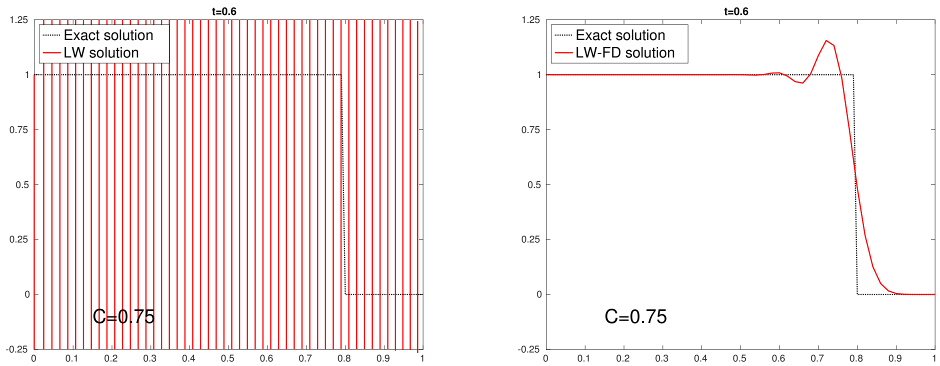

Figure 2: Comparison of the Lax-Wendroff scheme with (right) and without (left) a lumped mass matrix

Solve the problem using the third-order explicit Taylor-Galerkin method. Comment the results.

As a final step the third-order explicit Taylor-Galerkin method has been introduced and implemented in the MATLAB code. When comparing the result in Figure 3 with the solutions of the two preceding schemes a higher accuracy is notable. Only very small oscillations are present and the gradient at the discontinuity is closer to the exact solution.

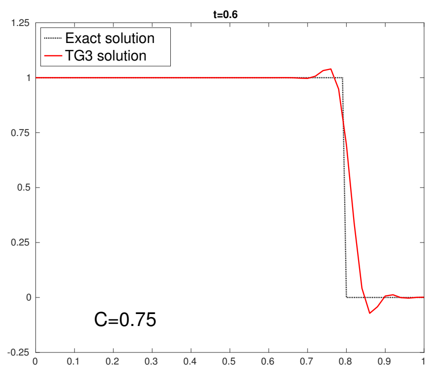

Figure 3: Results of the third-order explicit Taylor-Galerkin method.

References

[1] Jean Donea Antonio Huerta, *Finite Element Methods for Flow Problems*, Wiley, 2003.