Unsteady Navier-Stokes equation

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Introduction

One of the iconic equations when dealing with fluids is the so called Navier-Stokes equation. This equation is the resultant of applying physical conservation laws (such as linear momentum, angular momentum...), considering viscous effects, Newtonian behaviour and incompressibility. With that, it is possible to model a wide variety of problems such as ocean currents, atmospheric dynamics, air flow around a wing, a vehicle…

In this work, transient 2D cavity problem will be solved. This problem consist in a fluid inside a square box with all its walls fixed except the top one, which moves at velocity v=(v,0).
Since a viscous fluid is considered, no-slip condition must be imposed as boundary conditions.

Methodology

Navier-Stokes equation will be solved numerically by discretizing the equation using finite elements. The start point is the differential equation which reads:

Navier - Stokes equation

$$\left\{ \begin{array}{l} v_t + (v\cdot\nabla)v - \nu\nabla^2 v + \nabla p = f \\ \nabla\cdot v = 0 \end{array} \right\}$$

$$+$$

$$\left\{ \begin{array}{ll} v = v_D & \text{on } \Gamma_D \\ n\cdot\sigma = t & \text{on } \Gamma_N \end{array} \right\} \quad \text{B.C}$$

In this case, all boundary conditions are Dirichlet BC since no-slip condition implies velocities at the boundaries are prescribed. Because of this, only relative pressures can be obtained over the whole domain. This means that at least it is necessary to fix pressure at one point. In this work, pressure at the bottom left corner is set to zero.

With this considerations, Weighted Residuals method is applied to transform the differential equation to an integral form, also known as Weak form. Then Galerkin method is applied by choosing as test functions the shape functions used to interpolate the unknown function.

In this case, given that transient problem is being solved and a time derivative is involved. The approximation to the unknown function reads:

$$u \simeq u^h = \sum_{i=1}^{m} \begin{bmatrix} v_x(t) \\ v_y(t) \end{bmatrix} N(\vec{x})$$

It is important to notice that shape functions N(**x**) do not depend on time since time dependencies are imposed in the unknown values at nodes. With this consideration, the equation can be written in matrix form, after discretization as:

$$M\dot{u}(t) + [C(v)+K]u + Gp = f$$

where M is the mass matrix which comes from:

$$w\, u^h_t = [\mathrm{mat}\, N]\, u_t = [\mathrm{mat}\, N]^T \frac{\partial}{\partial t} [\mathrm{mat}\, N] \begin{bmatrix} v_x(t) \\ v_y(t) \end{bmatrix} = [\mathrm{mat}\, N][\mathrm{mat}\, N]\, \dot{u}$$

$$\text{identifying} \quad [\mathrm{mat}\, N]^T [\mathrm{mat}\, N] \equiv M = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ \vdots & \\ N_m & 0 \\ 0 & N_m \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_m & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_m \end{bmatrix} \equiv M$$

In the previous equations, **u** is the vector of nodal values.

At this point, different methods must be used to deal with time derivatives. In this report, Semi-Implicit method and Choric-Temam are going to be used.

Semi-implicit method

Semi-implicit method deals with the non-linear term C(v) and also the time derivative. Working, first, with the time derivative, θ-methods discretize this derivative with the following equation:

$$\frac{\Delta u}{\Delta t} - \theta \Delta u_t = u_t^n \qquad \text{with} \qquad \Delta u = u^{n+1} - u^n$$

using the following formula

$$M u_t + [C(v) + K] u + G p = f$$

isolating u_t:

$$\boxed{u_t = \frac{f - [C(v)+K]u - Gp}{M}}$$

It allows to write Δu_t and u_t in terms of C, K, G and M:

$$\frac{\Delta u}{\Delta t} - \theta\left[\frac{-(K+C^{n+1})u^{n+1} + (K+C^n)u^n - Gp^{n+1} + Gp^n}{M}\right] = \frac{f - (K+C^n)u^n - Gp^n}{M}$$

Now semi-implicit method consist in taking $C^{n+1}(v) = C^n(v)$ with that

$$\frac{\Delta u}{\Delta t} + \theta\left[\frac{(K+C^n)\Delta u + G\Delta p}{M}\right] = \frac{f - (K+C^n)u^n - Gp^n}{M}$$

arranging all terms Δu, Δp together and multiplying by M and Δt:

$$\boxed{[M + \theta \Delta t (K+C^n)]\Delta u + \Delta t\, \theta\, G \Delta p = \Delta t\,[f - (K+C^n)u^n - Gp^n]}$$

Finally, considering the incompressibility equation:

$$G^T \Delta u = 0$$

the system of equations can be written in matrix form as:

$$\begin{bmatrix} M + \theta \Delta t (K + C^n) & \Delta t \theta G \\ G & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta p \end{bmatrix} = \begin{bmatrix} \Delta t [f - (K + C^n) u^n - G p^n] \\ 0 \end{bmatrix}$$

Solving for Δu and Δp, unknown values can be obtained at each time step.

Chorin-Temam method

Chorin-Temam is a projection method which consist in computing the solution by using two steps. The first one, consist in computing an auxiliary velocity, independent of p, by just taking into account convective and viscous term and neglecting incompressibility condition.
Second step, consist in using this auxiliary velocity to compute velocity and pressure considering pressure term and also incompressibility condition.

The previous steps are stated mathematically in the following equations:

Step 1

$$M u_t + (K + C) u = f \qquad \text{No pressure term considered } (G p)$$

FE discretization yields

$$u^{int} = \frac{\Delta t f + M u^n}{M + \Delta t (K + C^n)}$$

$$\boxed{M + \Delta t (K + C^n) u^{int} = \Delta t f + M u^n}$$

here explicit method has been choosed ($v = v^n$) by computing convective term using $v = v^n$

With that intermediate velocity, step two reads:

Step 2

$$\begin{cases} M u_t + G p = 0 & \text{only pressure term considered} \\ \nabla \cdot u = 0 & \leftarrow \text{incompressibility condition} \end{cases}$$

First equation can be written as:

$$M u^{n+1} + \Delta t\, G p^{n+1} = M u^{int}$$

then, the system of equations reads: (in matrix form)

$$\begin{bmatrix} M & \Delta t\, G \\ G & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} M u^{int} \\ 0 \end{bmatrix}$$

with that, our system is discretized and velocity and pressure field can be obtained.

Results

Discretizing the square domain with a 10x10 mesh using Q2Q1 elements, for a fluid with Re=100 and ν=0.01, results have been obtained using different methods: semi-implicit with θ=1/2 (Crank-Nicolson), semi-implicit with θ=1 (implicit) and Chorin-Temam method.

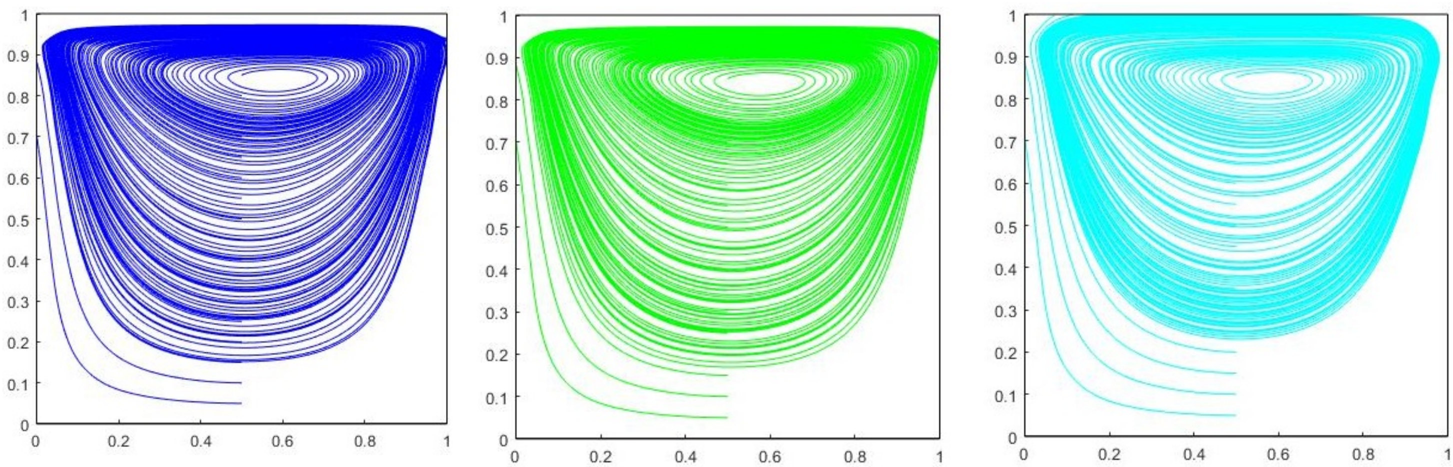

Figure 1. Streamlines at t=0.5s for θ=0.5 (left), θ=1 (mid) and Chorin-Temam (right).

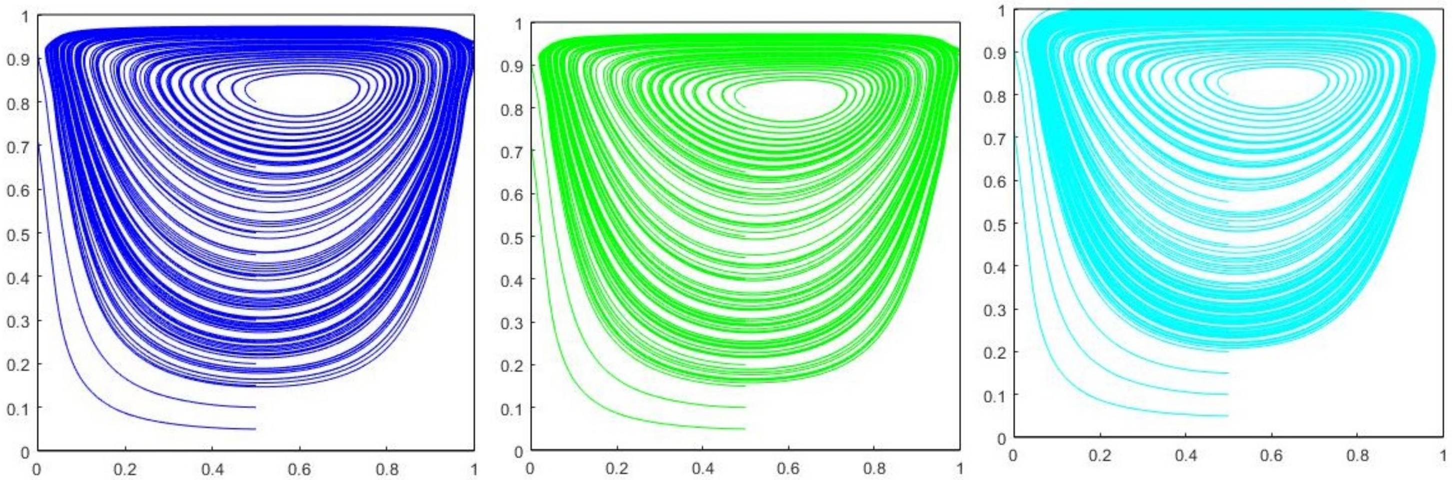

Figure 2. Streamlines at t=0.75s for θ=0.5 (left), θ=1 (mid) and Chorin-Temam (right).

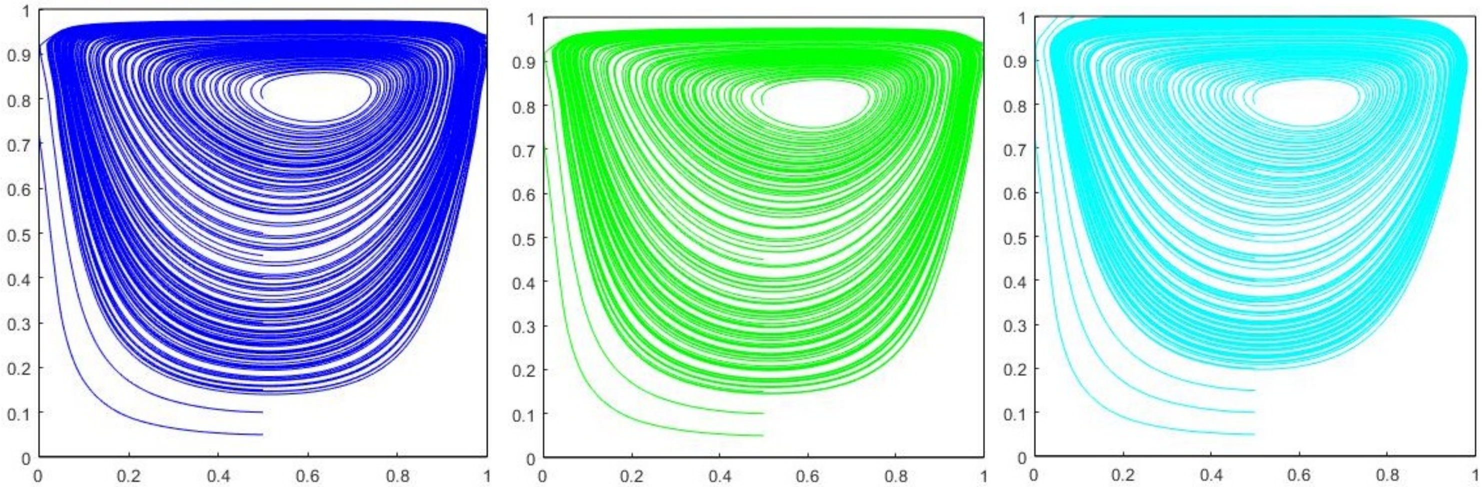

Figure 3. Streamlines at t=1s for θ=0.5 (left), θ=1 (mid) and Chorin-Temam (right).

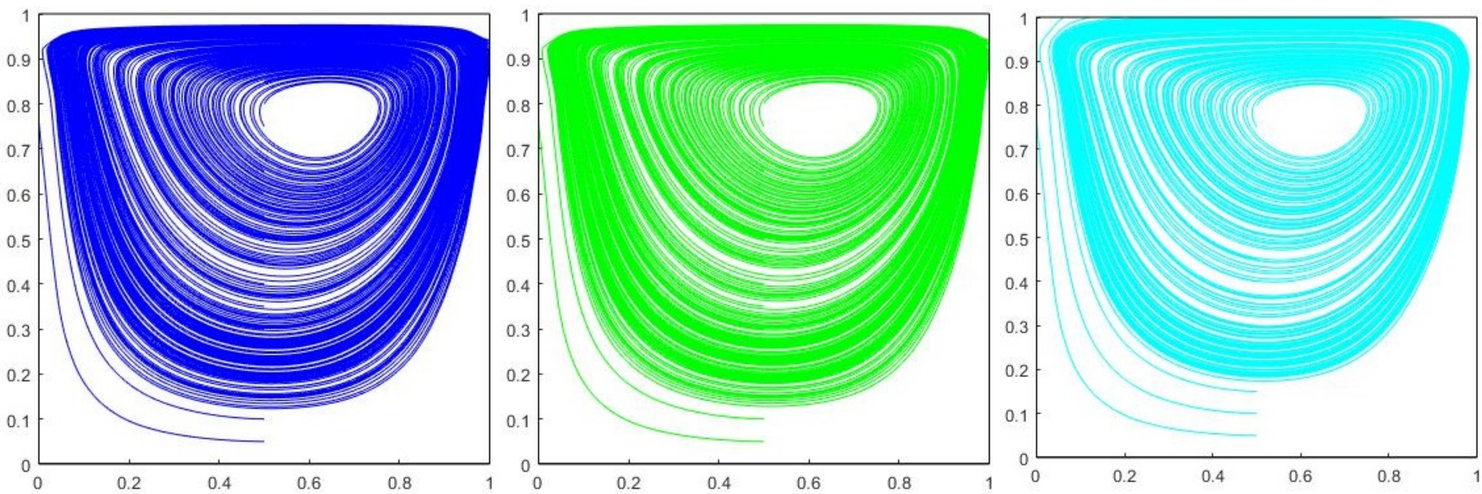

Figure 4. Streamlines at t=2s for θ=0.5 (left), θ=1 (mid) and Chorin-Temam (right).

Comparing all methods at different time steps, it is clearly seen that all methods provide a correct and similar solution. At first sight any remarkable difference between methods is appreciable.
It must also be remarked that, as in the case of non-transient Navier-Stokes equation, instabilities may happen if elements which doesn't fulfil LBB condition are used. Results using elements which don't fulfil this condition, Q1Q1 for instance, have not been used since they were studied in the

previous work. Also the effect of stabilization techniques, such as GLS, has not been shown in this work for the same reason.