Stokes numerical example

Problem formulation

Stokes problem

$$\begin{cases} -\nu\nabla^2\mathbf{v} + \nabla p = \mathbf{b} & \text{in } \Omega \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega \end{cases}$$

Weak form:

$$\begin{cases} \int_\Omega \nabla\mathbf{w} : \nu\nabla\mathbf{v}\, d\Omega - \int_\Omega p\nabla\cdot\mathbf{w}\, d\Omega = \int_\Omega \mathbf{w}\cdot\mathbf{b}\, d\Omega & \forall\mathbf{w}\in\mathbf{V} \\ \int_\Omega q\nabla\cdot\mathbf{v}\, d\Omega & \forall q\in\mathcal{Q} \end{cases}$$

Galerkin discretization:

$$\begin{pmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{h} \end{pmatrix}$$

Physical description

The boundary conditions for the present problem are presented in Figure 1. Except for the upper boundary all velocities are set to zero. On the upper edge the velocity in x_1-direction is set to one and in x_2-direction the velocity is zero again. This set up is also known as the cavity flow problem and hence an analytical solution exists it is widely used as benchmark for fluid computations. To give a physical example one could imagine a container filled with a fluid where one side is continuously moving similar to a conveyer belt.

Stability study

To assess the stability of the already implemented Galerkin discretization of the Stokes problem four different element types will be investigated:

- Q1Q1
- Q2Q1
- P1P1
- P2Q1

Figure 2 shows the streamlines obtained for the quadrilateral elements Q1Q1 (left) and Q2Q1 (right). At the first glance the streamlines for the Q1Q1 element show some irregularities. Looking at the corresponding pressure field in Figure 3 just underlines the impression that

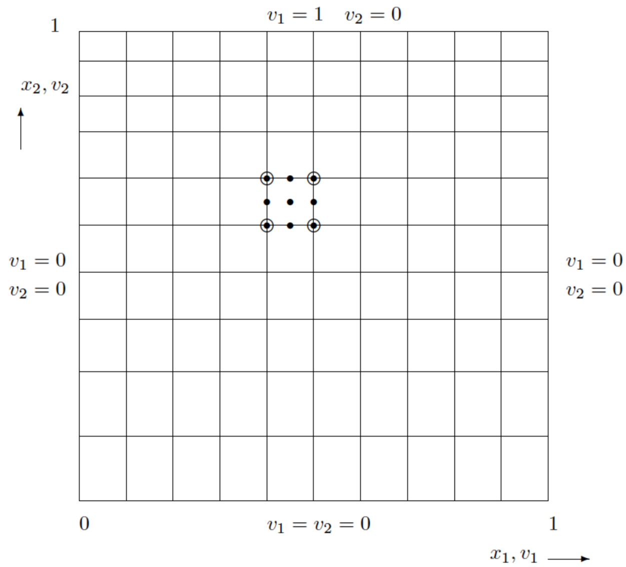

Figure 1: Boundary conditions of the cavity flow problem

the Q1Q1 is unstable. The Q2Q1 on the other hand produces smooth results without any stabilization technique applied.

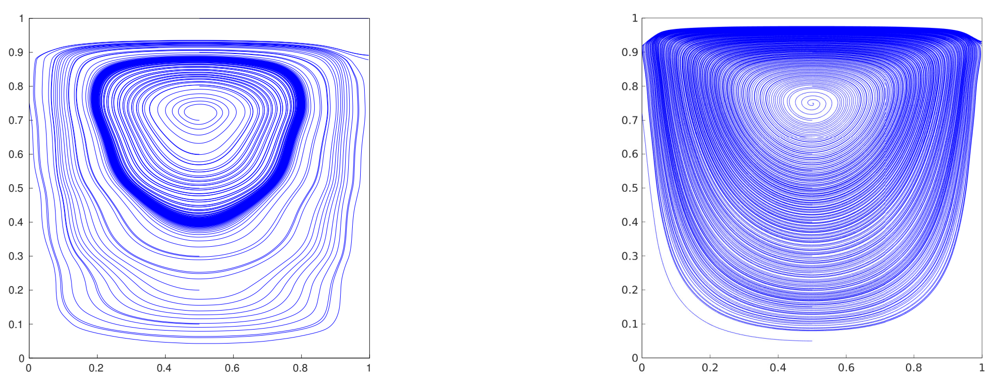

Figure 2: Streamlines for the Q1Q1 element (left) and the Q2Q1 element (right)

Figure 4 shows the streamlines and Figure 5 the pressure field for the triangular elements P1P1 (left) and P2P1 (right). Again, the same observation as for the quadrilateral elements can be made.

Stabilization method

One possibility to stabilize the Galerkin formulation is the use of the Galerkin/Least-squares method. The stabilization term and the discretization of the stabilized weak form were

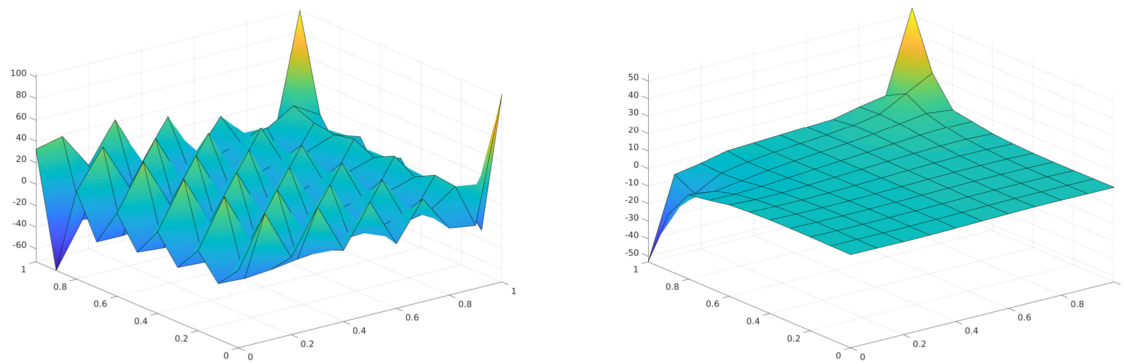

Figure 3: Pressure field for the Q1Q1 element (left) and the Q2Q1 element (right)

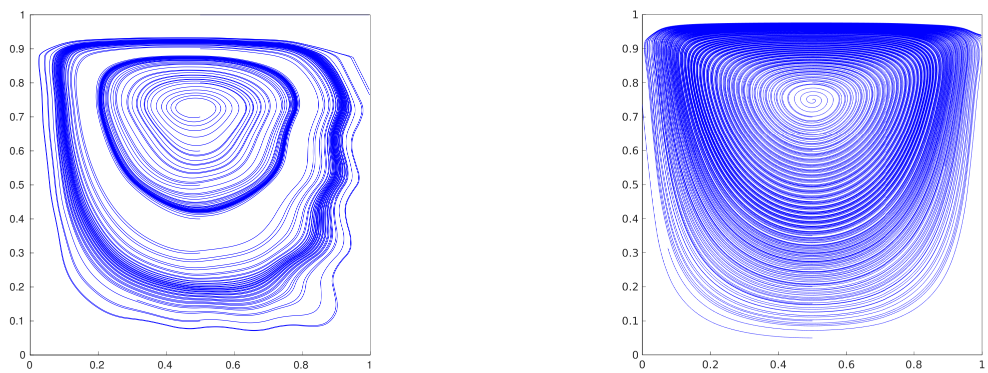

Figure 4: Streamlines for the P1P1 element (left) and the P2P1 element (right)

presented during the lecture. It can be written as followed:

$$\begin{pmatrix} \mathbf{K}+\bar{\mathbf{K}} & \mathbf{G}+\bar{\mathbf{G}} \\ \mathbf{G}^T+\bar{\mathbf{G}}^T & \mathbf{0}+\bar{\mathbf{L}} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f}+\bar{\mathbf{f}}_w \\ \mathbf{h}+\bar{\mathbf{f}}_q \end{pmatrix}$$

As seen in the previous examples a stabilization technique is only needed for element types with a velocity discretization of degree one. In this case the matrix problem simplifies to

$$\begin{pmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & \bar{\mathbf{L}} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \bar{\mathbf{f}}_q \end{pmatrix}$$

with

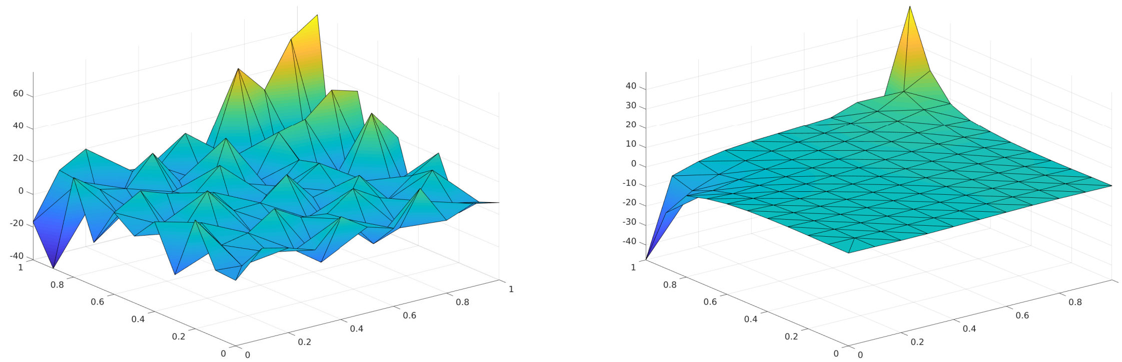

Figure 5: Pressure field for the P1P1 element (left) and the P2P1 element (right)

$$\bar{\mathbf{L}} = \sum_e \int_{\Omega_e} \tau_1 (\nabla q) \cdot (\nabla p) \, d\Omega$$
$$\bar{\mathbf{f}}_q = \sum_e \int_{\Omega_e} \tau_1 (\nabla q) \cdot (-\mathbf{f}) \, d\Omega$$

and

$$\tau_1 = \alpha_0 \frac{h^2}{4\nu}, \quad \alpha_0 = \frac{1}{3} \text{ (optimal for linear elements).}$$

In order to implement $\bar{\mathbf{L}}$ in the existing MATLAB code we can follow a similar approach as for the discretization of the matrix $\mathbf{K}$. One difference is that on the element level $\bar{\mathbf{L}}$ is only a $[3 \times 3]$ matrix whereas $\mathbf{K}$ is of dimensions $[6 \times 6]$. Thus, the reshaping step is not necessary. Eventually, the matrix can be expressed as the outer product of the shape function vectors defined in global coordinates. For the implementation of $\bar{\mathbf{f}}_q$ the source term vector $\mathbf{f}$ has to be simply multiplied by the same shape function vectors. For both cases τ_1 is defined as scalar value and is simply inserted as multiplier in the assembly loop.

To asses the correctness of the implementation the computation of the cavity flow problem was repeated for the quadrilateral and the triangular element with a velocity discretization of order one. The results are shown in Figure 6 and Figure 7. For both element types stable solutions are obtained. However, when compared with the higher order elements a stronger smoothing caused by the GLS stabilization method is notable.

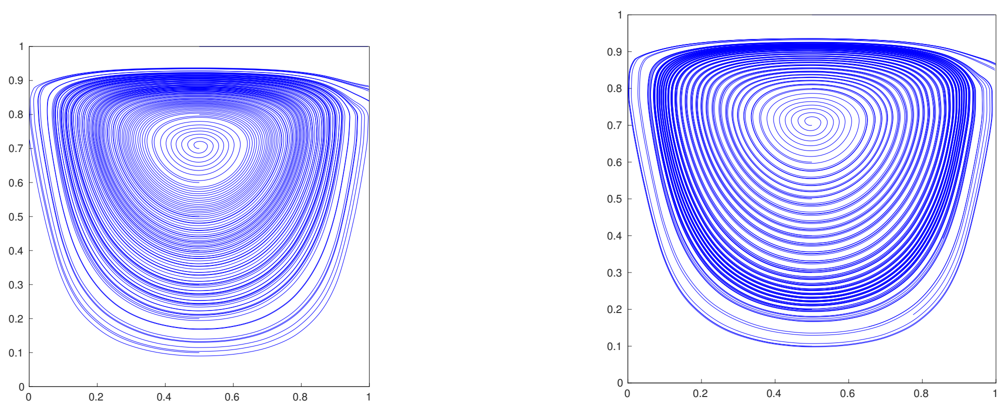

Figure 6: Streamlines for the stabilized elements Q1Q1 (left) and P1P1 (right)

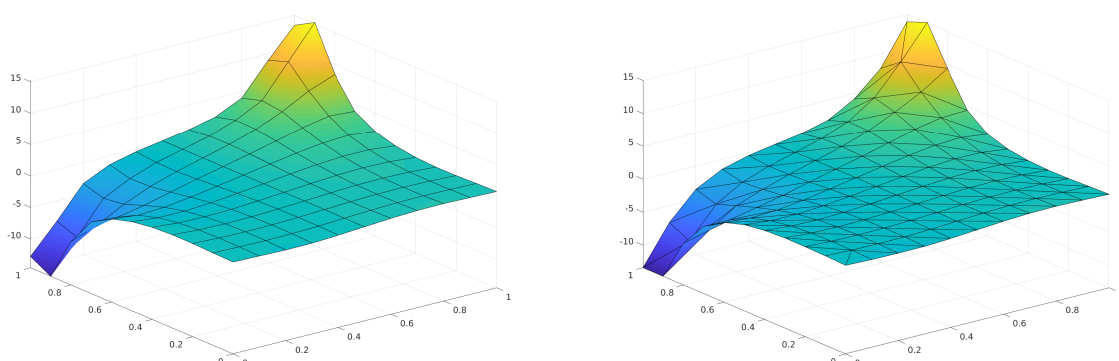

Figure 7: Pressure field for the stabilized elements Q1Q1 (left) and P1P1 (right)