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Assignement 3:

→ Write the one-step and two-steps Taylor-Galerkin method for the perturbated Burger's equation:

$$\begin{cases} u_t + u u_x = \varepsilon u_{xx} & , \text{for } (x;t) \in [-1;1] \times [0,T] \\ u(x;0) = u_0(x) & , \text{for } x \in [-1;1] \\ u(-1;t) = u(1,t) = 0 & , \text{for } t \in [0;T] \end{cases}$$

a) One-step Taylor-Galerkin:

The second-order expansion in time is written: $u^{n+1} = u^n + \Delta t\, u_t^n + \frac{1}{2} \Delta t^2 u_{tt}^n$

But thanks to the perturbated Burgers' equation: $u_t = -\underbrace{u u_x}_{f_x} + \varepsilon u_{xx}$

So:

$$u_t = -f_x + \varepsilon u_{xx}$$

$$\begin{aligned} u_{tt} &= -f_{xt} + \varepsilon u_{xxt} = -f_{tx} + \varepsilon u_{txx} \\ &= -(f_u u_t)_x + \varepsilon (u_t)_{xx} \\ &= -(f_u u_t)_x + \varepsilon(-f_x + \varepsilon u_{xx})_{xx} \\ &= -(f_u u_t)_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx}) \end{aligned}$$

Thus, substituting u_t and u_{tt} in the Taylor expansion:

$$\frac{u^{n+1} - u^n}{\Delta t} = -f_x + \varepsilon u_{xx} + \frac{\Delta t}{2}\left(\left(-u f_x + \varepsilon u u_{xx}\right)_x + \varepsilon\left(-f_{xxx} + \varepsilon u_{xxxx}\right)\right)$$

multiplied by the test function ω and integrating over the Ω domain:

$$\int_\Omega \omega \frac{u^{n+1} - u^n}{\Delta t} = \int_\Omega \omega (-f_x + \varepsilon u_{xx}) dx + \frac{\Delta t}{2} \int_\Omega \omega \left[(u f_x - \varepsilon u u_{xx})_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx}) \right] dx$$

We then integrate by parts:

$$\int_{-1}^{1} \omega \frac{u^{n+1} - u^n}{\Delta t} dx = \int_{-1}^{1} \omega_x (f - \varepsilon u_x) dx + \frac{\Delta t}{2} \left[-\int_{-1}^{1} \omega_x \left(u(f_x - \varepsilon u_{xx}) - \varepsilon(f_{xx} - \varepsilon u_{xxx}) \right) dx \right]$$

$$+ \left[\omega(-f + \varepsilon u_x) \right]_{-1}^{1} + \frac{\Delta t}{2} \omega \left[u(f - \varepsilon u_{xx}) - \varepsilon(f_{xx} - \varepsilon u_{xxx}) \right]_{-1}^{1}$$

b) Two-step Taylor-Galerkine

$$\begin{cases} u^{n+\frac{1}{2}} = u^n + \dfrac{\Delta t}{2} u_t^n & (1) \\ u^{n+1} = u^n + \Delta t \, u_t^{n+\frac{1}{2}} & (2) \end{cases}$$

We substitute $u_t = -f_x + \varepsilon u_{xx}$ in (2) : $\dfrac{u^{n+1} - u^n}{\Delta t} = (-f_x + \varepsilon u_{xx})^{n+\frac{1}{2}}$

multiplied by w and integrating:

$$\int_{-1}^{1} w \frac{u^{n+1} - u^n}{\Delta t} dx = \int_{-1}^{1} w \left(-f_x + \varepsilon u_{xx}\right)^{n+\frac{1}{2}} dx = \left[w \left(-f^{n+\frac{1}{2}} + \varepsilon u_x^{n+\frac{1}{2}}\right) \right]_{-1}^{1} - \int_{-1}^{1} w_x \left(-f^{n+\frac{1}{2}} + \varepsilon u_x^{n+\frac{1}{2}}\right) dx$$

Values inside the integrals are ~~have~~ evaluated at an intermediate point given by (1)