Ye Mao HW-3

one step:

$$u_t = -\underbrace{uu_x}_{f_x} + \varepsilon u_{xx}$$

this problem we consider u as u^ε

let $uu_x = f_x \Rightarrow f = \frac{u^2}{2}$ $\quad f_u = u$

$$u_t = -f_x + \varepsilon u_{xx}$$

$$u_{tt} = -f_{xt} + \varepsilon u_{xxt} = -f_{tx} + \varepsilon u_{txx}$$

$$-(f_u u_t)_x + \varepsilon(-f_x + \varepsilon u_{xx})_{xx} = -(f_u u_t)_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx})$$

$$= -[f_u(-f_x + \varepsilon u_{xx})]_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx})$$

$$= -(-uf_x + \varepsilon u u_{xx})_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx})$$

second-order Taylor expansion of time

$$u^{n+1} = u^n + \Delta t u_t^n + \frac{\Delta t^2}{2} u_{tt}^n$$

$$\frac{u^{n+1} - u^n}{\Delta t} = u_t^n + \frac{\Delta t}{2} u_{tt}^n$$

$$= -f_x + \varepsilon u_{xx} + \frac{\Delta t}{2}[-(-uf_x + \varepsilon u u_{xx})_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx})$$

multiplying by test function w, and integrating over the domain

$$\int_\Omega w \frac{u^{n+1} - u^n}{\Delta t} dx = \int_\Omega w(-f_x + \varepsilon u_{xx})dx + \frac{\Delta t}{2}\int_\Omega w[(uf_x - \varepsilon u u_{xx})_x + \varepsilon(-f_{xxx} + \varepsilon u_{xxxx})] dx$$

$$= -wf\big|_\Omega + \int_\Omega w_x f dx + \varepsilon w u_x\big|_\Omega - \int \varepsilon u_x w_x dx$$

$$+ \frac{\Delta t}{2}\Big[wu(f_x - \varepsilon u_{xx})\big|_\Omega - \int_\Omega w_x u(f_x - \varepsilon u_{xx})dx$$

$$+ \varepsilon w(-f_{xx} + \varepsilon u_{xxx})\big|_\Omega - \varepsilon\int_\Omega w_x(-f_{xx} + \varepsilon u_{xxx})dx$$

$$\int_{-1}^{1} w\frac{u^{n+1} - u^n}{\Delta t} dx = \int_{-1}^{1} w_x(f - \varepsilon u_x)dx + w(-f + \varepsilon u_x)\big|_{-1}^{1} + \frac{\Delta t}{2} w[u(f - \varepsilon u_{xx})$$

$$- \varepsilon(f_{xx} - u_{xxx})]_{-1}^{1} + \frac{\Delta t}{2}\Big\{-\int_{-1}^{1} w_x[u(f_x - \varepsilon u_{xx}) - \varepsilon(f_{xx} - \varepsilon u_{xxx})]dx\Big\}$$

two step:

$$\begin{cases} u^{n+1/2} = u^n + \frac{\Delta t}{2} u_t^n & ① \\ u^{n+1} = u^n + \Delta t\, u_t^{n+1/2} & ② \end{cases}$$

since $u_t = -f_x + \varepsilon u_{xx}$

from ② yields $\frac{u^{n+1} - u^n}{\Delta t} = u_t^{n+1/2} = (-f_x + \varepsilon u_{xx})^{n+1/2}$

multiplying by w and integrating:

$$\int_{-1}^{1} w \frac{u^{n+1} - u^n}{\Delta t} dx = \int_{-1}^{1} w(-f_x + \varepsilon u_{xx})^{n+1/2} dx$$

$$= w(-f + \varepsilon u_x)^{n+1/2} \Big|_{-1}^{1} - \int_{-1}^{1} w_x (-f_x + \varepsilon u_x)^{n+1/2} dx$$

here the values on the integration with $(n+1/2)$ superscript are evaluated at the intermediary point which is given by equation ①