Finite Elements in Fluids

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We consider the pertubed Burgers' equation with **strong form**

$$\begin{aligned} &u_t + uu_x = \epsilon u_{xx} && \text{for } (x,t) \in [-1,1] \times [0,T] \\ &u(x,0) = u_0(x) && \text{for } x \in [-1,1] \\ &u(-1,t) = u(1,t) = 0 && \text{for } T \in [0,T] \end{aligned}$$

So $f(u) = \frac{u^2}{2} - \epsilon u_x$ and $a(u) := \frac{\partial f}{\partial u} = u$.

1 One-step Taylor-Galerkin method

Considering $u^n := u^n(x) = u(x, n\Delta t)$,

$$\begin{aligned} u^{n+1} &= u^n - \Delta t f_x(u^n) + \frac{(\Delta t)^2}{2}(a(u^n) f_x(u^n))_x \\ &= u^n - \Delta t(u^n u^n_x - \epsilon u^n_{xx}) + \frac{(\Delta t)^2}{2}(u^n \cdot (u^n u^n_x - \epsilon u^n_{xx}))_x \end{aligned}$$

1.1 Weak form

Consider $\omega \in H^1_0([-1,1]) = \{v \in H^1([-1,1]) : v(-1) = v(1) = 0\}$,

$$\begin{aligned} \int_{-1}^1 u^{n+1}\omega dx &= \int_{-1}^1 (u^n - \Delta t(u^n u^n_x - \epsilon u^n_{xx}) + \frac{(\Delta t)^2}{2}(u^n \cdot (u^n u^n_x - \epsilon u^n_{xx}))_x)\omega dx \\ &= \int_{-1}^1 u^n \omega dx - \Delta t \int_{-1}^1 (u^n u^n_x - \epsilon u^n_{xx})\omega dx + \frac{(\Delta t)^2}{2} \int_{-1}^1 (u^n \cdot (u^n u^n_x - \epsilon u^n_{xx}))_x \omega dx \end{aligned}$$

Considering the integration by part formula,

$$\begin{aligned} \int_{-1}^1 f_x(u^n)\omega dx &= [f(u^n(1))\omega(1) - f(u^n(-1))\omega(-1)] - \int_{-1}^1 f(u^n)\omega_x dx \\ &= -\int_{-1}^1 (\frac{(u^n)^2}{2} - \epsilon u^n_x)\omega_x dx \\ \int_{-1}^1 (a(u^n) f_x(u^n))_x \omega dx &= [a(u^n(1)) f_x(u^n(1))\omega(1) - a(u^n(-1)) f_x(u^n(-1))\omega(-1)] - \int_{-1}^1 (a(u^n) f_x(u^n))\omega_x dx \\ &= -\int_{-1}^1 (u^n \cdot (u^n u^n_x - \epsilon u^n_{xx}))\omega_x dx \end{aligned}$$

Thus, the **weak form** of the pertubed Burgers' equation is:

Find $u^{n+1} \in H^1([-1,1])$ such that

$$\int_{-1}^{1} u^{n+1}\omega dx = \int_{-1}^{1} u^n \omega dx + \Delta t \int_{-1}^{1} (\frac{(u^n)^2}{2} - \epsilon u_x^n)\omega_x dx - \frac{(\Delta t)^2}{2}\int_{-1}^{1}(u^n \cdot (u^n u_x^n - \epsilon u_{xx}^n))\omega_x dx \qquad (1)$$

$\forall \omega \in H_0^1([-1,1])$.

1.2 FE Discretization

We consider the set of basis functions $\{N_i(x)\}_{i=1,\dots,m}$ of $H^1([-1,1])$, we suppose $x_1 = -1$ and $x_m = 1$. We define as our approximate solutions $u(x,(n+1)\Delta t) \approx u^h(x,(n+1)\Delta t) = \sum_{i=1}^{m} u_i^{n+1} N_i(x)$ where $u_i^{n+1} = u(x_i,(n+1)\Delta t)$. We also consider $\omega = \sum_{i=2}^{m-1} \omega_i N_i(x)$ since $\omega \in H_0^1([-1,1])$, so $\omega(-1) = \omega_1 = 0$ and $\omega(1) = \omega_m = 0$.

Thus, substituting in equation 1, we have,

$$\mathbf{W}^T (M)_{i=2,\dots,m-1,j=1,\dots,m} \mathbf{U}^{n+1} = \mathbf{W}^T (F)_{i=1,\dots m}$$

where $\mathbf{W} = (\omega_2, \dots, \omega_{m-1})^T$, $\mathbf{U}^n = (u_1^{n+1}, \dots, u_m^{n+1})^T$, $M_{ij} = \int_{-1}^{1} N_i(x)N_j(x)dx$ is the mas matrix and $F_i = \int_{-1}^{1} u^n N_i(x)dx + \Delta t \int_{-1}^{1}(\frac{(u^n)^2}{2} - \epsilon u_x^n)(N_i)_x(x)dx - \frac{(\Delta t)^2}{2}\int_{-1}^{1}(u^n \cdot (u^n u_x^n - \epsilon u_{xx}^n))(N_i)_x(x)dx$.

As the weak form has to be accomplished $\forall \omega \in H_0^1([-1,1])$, we can eliminate it from our equation. From the Dirichlet boundary conditions, we know $u_1^n = u(-1, n\Delta t) = 0$ and $u_m^n = u(1, n\Delta t) = 0$. So at each time iteration we have to solve the following system:

$$M_{i,j=2,\dots,m-1} \mathbf{U}_{i=2,\dots,m-1}^{n+1} = \mathbf{F}_{i=2,\dots,m-1}$$

2 Two-step Taylor-Galerkin method

In this case we will use he following scheme:

$$\begin{aligned} u^{n+1/2} &= u^n - \frac{\Delta t}{2} f_x(u^n) \\ u^{n+1} &= u^n - \Delta t f_x(u^{n+1/2}) \end{aligned}$$

2.1 Weak form

As in the One-step method, we compute the weak form of the problem for both equations,

$$\begin{aligned} \int_{-1}^{1} u^{n+1/2}\omega dx &= \int_{-1}^{1} u^n \omega dx - \frac{\Delta t}{2}\int_{-1}^{1} f_x(u^n)\omega dx \\ \int_{-1}^{1} u^{n+1}\omega dx &= \int_{-1}^{1} u^n \omega dx - \Delta t \int_{-1}^{1} f_x(u^{n+1/2})\omega dx \end{aligned}$$

Using the interation by parts formula,

$$\begin{aligned}\int_{-1}^{1} f_x(u^n)\omega dx &= [f(u^n(1))\omega(1) - f(u^n(-1))\omega(-1)] - \int_{-1}^{1} f(u^n)\omega_x dx = -\int_{-1}^{1} (\frac{(u^n)^2}{2} - \epsilon u_x^n)\omega_x dx \\ \int_{-1}^{1} f_x(u^{n+1/2})\omega dx &= -\int_{-1}^{1} (\frac{(u^{n+1/2})^2}{2} - \epsilon u_x^{n+1/2})\omega_x dx\end{aligned}$$

So the **weak form** of each equation will be:

Find $u^{n+1}, u^{n+1/2} \in H^1([-1,1])$ such that

$$\begin{aligned}\int_{-1}^{1} u^{n+1/2}\omega dx &= \int_{-1}^{1} u^n \omega dx + \frac{\Delta t}{2}\int_{-1}^{1} (\frac{(u^n)^2}{2} - \epsilon u_x^n)\omega_x dx \\ \int_{-1}^{1} u^{n+1}\psi dx &= \int_{-1}^{1} u^n \psi dx + \Delta t \int_{-1}^{1} (\frac{(u^{n+1/2})^2}{2} - \epsilon u_x^{n+1/2})\psi_x dx\end{aligned}$$

$\forall \omega, \psi \in H_0^1([-1,1])$.

2.2 FE Discretization

As we did in the One-step method, we discretize our domain and we end with a linear system to solve for each equation:

$$M_{i,j=2,\ldots,m-1}\mathbf{U}_{i=2,\ldots,m-1}^{n+1/2} = \mathbf{G}_{i=2,\ldots,m-1} \quad (2)$$

$$M_{i,j=2,\ldots,m-1}\mathbf{U}_{i=2,\ldots,m-1}^{n+1} = \mathbf{H}_{i=2,\ldots,m-1} \quad (3)$$

where $\mathbf{G}_i = \int_{-1}^{1} u^n N_i(x)dx + \frac{\Delta t}{2}\int_{-1}^{1}(\frac{(u^n)^2}{2} - \epsilon u_x^n)(N_i)_x(x)dx$ and $\mathbf{H}_i = \int_{-1}^{1} u^n N_i(x)dx + \Delta t\int_{-1}^{1}(\frac{(u^{n+1/2})^2}{2} - \epsilon u_x^{n+1/2})(N_i)_x(x)dx$.

So at each time iteration we will have to compute the solution of equation 2 to evaluate $\mathbf{H}$ and find the solution of 3.