Master Of Science in Computational Mechanics

Finite Elements in Fluid

Chinmay Khisti

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Assignment 4

Unsteady convection-Propagation of a steep front

The problem studied consist of unsteady convection equation with following initial and boundary conditions.

$$\begin{cases} u_t + au_x = 0 & x \in (0,1), t \in (0,0.6] \\ u(x,0) = u_0 & x \in (0,1) \\ u(0,t) = 1 & t \in (0,0.6] \end{cases}$$

$$u_0 = \begin{cases} 1 & \text{if } x \leq 0.2 \\ 0 & \text{otherwise} \end{cases}$$

As per given data, the solution will be computed using a convection velocity of $a = 1$, discretization in space with size $h = \Delta x = 2X10^{-2}$ and in time $\Delta t = 1.5X10^{-2}$. Therefore, the **Courant number comes out to be 0:75**.

Now, the code provided can fully compute Crank Nicolson time scheme with galerkin formulation in space with consistent mass matrix. to see the effect of lumped mass matrix in the solution following equations were added to already available $FEM_matrices.m$ file.

$$M_{ij}^{lump} = \begin{cases} \int_\Omega N_i N_j d\Omega & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

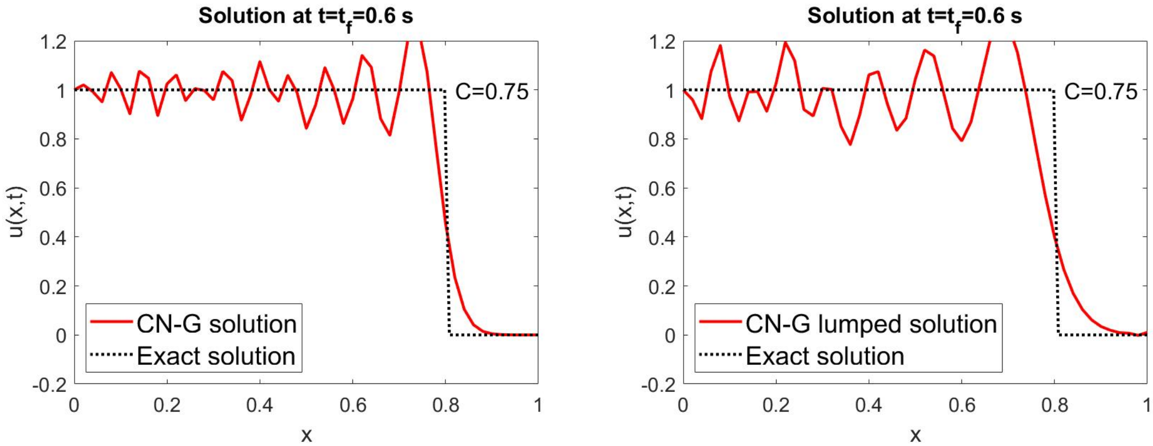
Following equations were used to define various matrices

$$M- : \quad M_{ij} = \int_\Omega N_i N_j d\Omega \quad \text{Consistent mass matrix}$$

$$C- : \quad C_{ij} = \int_\Omega N_i (a.\nabla N_j) d\Omega \quad \text{Convection matrix}$$

$$K- : \quad K_{ij} = \int_\Omega (\nabla N_i . \nabla N_j) d\Omega \quad \text{Stiffness matrix}$$

Crank-Nicholson scheme in time and linear finite element for the Galerkin scheme in space: As the solution in the figure 1. suggest that both the solutions with consistent and lumped mass matrix are considerably stable with minor oscillations.But, solution with consistent mass matrix figure 1a gives better result then Lumped mass matrix figure 1b.



(a) Consistent mass matrix.

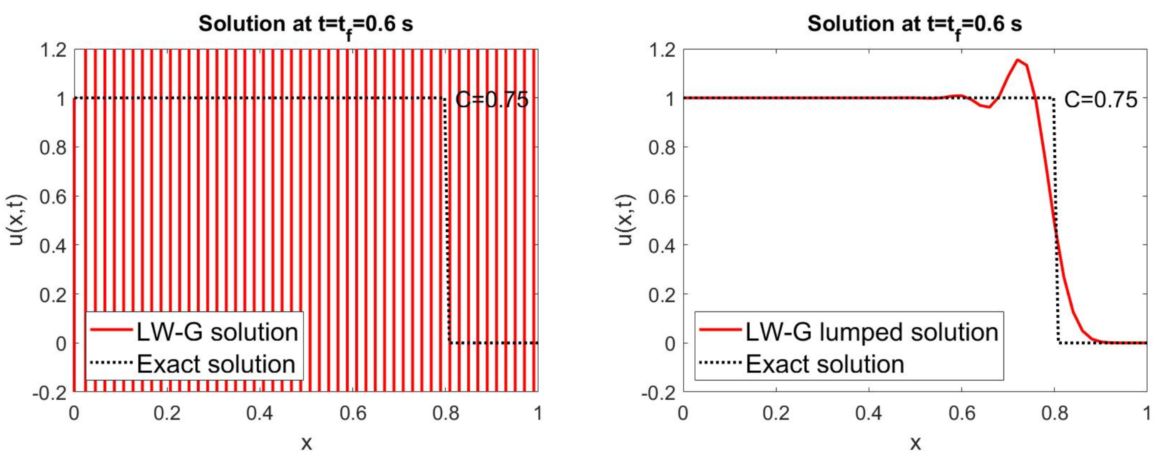
(b) Lumped mass matrix.

Figure 1: Crank-Nicholson scheme

Lax-Wendroff scheme in time and Galerkin scheme in space: For implementation of this scheme equation 1 is introduced in the code for consistent and Lumped mass matrix,

$$\begin{aligned} &\text{Lax Wendroff with consistent mass matrix} \quad \tfrac{1}{\Delta t}M = f + Cu^n - \tfrac{\Delta t}{2}||a||^2 Ku^n \\ &\text{Lax Wendroff with Lumped mass matrix} \quad \tfrac{1}{\Delta t}M^{lump} = f + Cu^n - \tfrac{\Delta t}{2}||a||^2 Ku^n \end{aligned} \tag{1}$$

Figure 2 clearly lies on the path of theory stating the stability range $C <\approx .0.577$, Figure 2a with C = 0.75 is completely in stable while after implementing Lumped mass matrix the solution improves significantly as seen in the figure 2b.



(a) Consistent mass matrix.

(b) Lumped mass matrix.

Figure 2: Lax-Wendroff scheme

Third-order explicit Taylor-Galerkin: For implementation of this scheme equation 2 is introduced in the code,

$$(\frac{1}{\Delta t}M + \frac{\Delta t}{6}||a||^2 K)\Delta u = f + Cu^n - \frac{\Delta t}{2}||a||^2 Ku^n \tag{2}$$

The figure 3 shows solution for the scheme, as it is third order in time it is much more stable then Crank-Nicholson and Lax-Wendroff scheme. specially for $c \leq 1$.

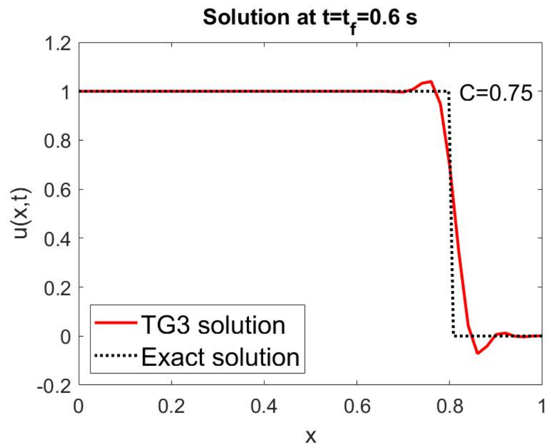


Figure 3: Third-order Taylor Galerkin.