FEF-HW5: Stokes and Navier-Stokes equations

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1 Stokes problem

The **strong form** of the Stokes problem is:

$$\begin{aligned} -\nu\nabla^2\mathbf{v} + \nabla p &= \mathbf{b} \ \text{ in } \Omega \\ \nabla\cdot\mathbf{v} &= 0 \ \text{ in } \Omega \end{aligned}$$

Considering $\omega \in \mathcal{V}$ and $q \in \mathcal{Q}$, we end with the **weak form**:

$$\begin{aligned} \int_\Omega \nabla\omega : \nu\nabla\mathbf{v}\, d\Omega - \int_\Omega p\nabla\cdot\omega\, d\Omega &= \int_\Omega \omega\cdot\mathbf{b}\, d\Omega \\ \int_\Omega q\nabla\cdot\mathbf{v}\, d\Omega &= 0 \end{aligned}$$

Considering $\{N_i(\mathbf{x})\}$ a base of $\mathcal{V}$ and $\{\tilde{N}_i(\mathbf{x})\}$ a base of $\mathcal{Q}$, we approximate our functions by

$$\begin{aligned} \mathbf{v} \approx \mathbf{v}^h &= \sum_j \mathbf{v_j} N_j(\mathbf{x}) \\ p \approx p^h &= \sum_k p_k \tilde{N}_k(\mathbf{x}) \end{aligned}$$

and we define the auxiliar functions ω and q by

$$\begin{aligned} \omega &= \sum_i \omega_{\mathbf{i}} N_i(\mathbf{x}) \\ q &= \sum_l q_l \tilde{N}_l(\mathbf{x}) \end{aligned}$$

Substituting in the weak form,

$$\begin{aligned} \mathbf{W}^T \int_\Omega (grad\,\mathbf{N})^T \nu (grad\,\mathbf{N})\, d\Omega\, \mathbf{V} - \mathbf{W}^T \int_\Omega \mathbf{D}^T (mat\,\tilde{\mathbf{N}})\, d\Omega\, \mathbf{P} &= \mathbf{W}^T \int_\Omega (mat\,\mathbf{N})\cdot\mathbf{b}\, d\Omega \\ \mathbf{Q}^T \int_\Omega (mat\,\tilde{\mathbf{N}})^T \mathbf{D}\, d\Omega\, \mathbf{V} &= 0 \end{aligned}$$

where

$$
\begin{aligned}
(grad\,\mathbf{N}) &= \begin{pmatrix} \frac{\partial N_1}{\partial x} & 0 & \dots & \frac{\partial N_m}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial x} & \dots & 0 & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_1}{\partial y} & 0 & \dots & \frac{\partial N_m}{\partial y} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \dots & 0 & \frac{\partial N_m}{\partial y} \end{pmatrix} \\
(mat\,\mathbf{N}) &= \begin{pmatrix} N_1 & 0 & \dots & N_m & 0 \\ 0 & N_1 & \dots & 0 & N_m \end{pmatrix} \\
(mat\,\tilde{\mathbf{N}}) &= \begin{pmatrix} \tilde{N_1} & \tilde{N_2} & \dots & \tilde{N_m} \end{pmatrix} \\
(grad\,\tilde{\mathbf{N}}) &= \begin{pmatrix} \frac{\partial \tilde{N_1}}{\partial x} & \frac{\partial \tilde{N_2}}{\partial x} & \dots & \frac{\partial \tilde{N_m}}{\partial x} \\ \frac{\partial \tilde{N_1}}{\partial y} & \frac{\partial \tilde{N_2}}{\partial y} & \dots & \frac{\partial \tilde{N_m}}{\partial y} \end{pmatrix} \\
\mathbf{D} &= [1,0,0,1](grad\,\mathbf{N}) \qquad (1) \\
\mathbf{V}^T &= [V_{x1}, V_{y1}, ...V_{xm}, V_{ym}] \qquad (2) \\
\mathbf{P}^T &= [P_1, P_2, ...P_{m-1}, P_m] \qquad (3)
\end{aligned}
$$

Changing the sign of the last equation we end with the linear system:

$$
\begin{pmatrix} \mathbf{K} & \mathbf{G}^T \\ \mathbf{G} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix} \qquad (4)
$$

where

$$
\mathbf{K} = \int_\Omega [grad\,\mathbf{N}]^T \nu [grad\,\mathbf{N}]\, d\Omega
$$

$$
\mathbf{G} = -\int_\Omega [mat\,\tilde{\mathbf{N}}]^T D\, d\Omega
$$

$$
\mathbf{f} = \int_\Omega [mat\,\mathbf{N}] \cdot \mathbf{b}\, d\Omega
$$

1.1 Results

Considering different basis for the two spaces $\mathcal{V}$ and $\mathcal{Q}$, we obtain the following results, that are represented at Fig. 1.

Type of element for $\mathcal{V}$	Q1	Q2	P1	P2
Type of element for $\mathcal{Q}$	Q1	Q1	P1	P1
Stable?	No	Yes	No	Yes

Table 1: Where Q denotes a quadrilateral element and P a triangle element, and the number specifies the degree of the functions $N/\tilde{N}$.

As we can see, we need to stabilize the results for linear elements for $\mathcal{V}$.

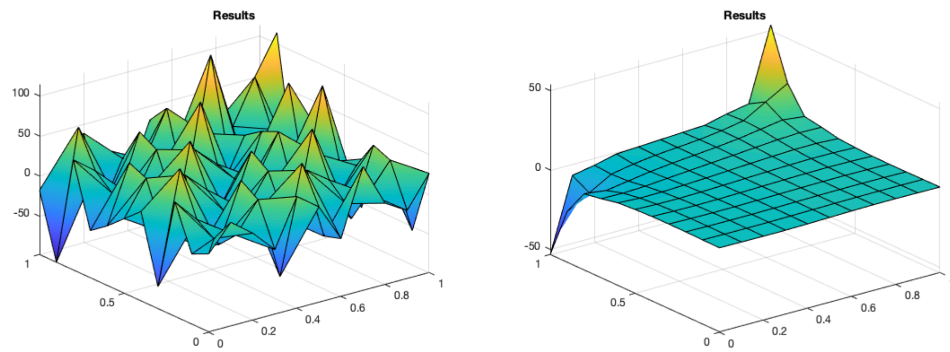

Figure 1: Examples of the resulting pressure. On the left, results for P1P1 elements, and on the right, results for Q2Q1 elements.

1.2 Stabilization method (GLS)

We consider the GLS stabilization method which consists in adding stabilization terms to the weak form:

$$\sum_e \int_{\Omega_e} \tau \mathcal{L}(\omega, q)(\mathcal{L}(\mathbf{v}, p) - \mathcal{F})\, d\Omega \tag{5}$$

where

$$\mathcal{L}(\mathbf{v}, p) = \begin{bmatrix} -\nu \nabla^2 \mathbf{v} + \nabla p \\ \nabla \cdot \mathbf{v} \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

Adding the term to the weak form and using that we are only stabilizing for linear element in velocity and in pressure (so $\nabla^2 \mathbf{v} = \nabla^2 \omega = 0$), we end with the system

$$\begin{pmatrix} \mathbf{K} + \bar{\mathbf{K}} & \mathbf{G}^T \\ \mathbf{G} & \bar{\mathbf{L}} \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \bar{\mathbf{f}}_{\mathbf{q}} \end{pmatrix}$$

where

$$\begin{aligned} \bar{\mathbf{K}} &= \sum_e \int_{\Omega_e} \tau_2 \mathbf{D}^T \mathbf{D}\, d\Omega \\ \bar{\mathbf{L}} &= \sum_e \int_{\Omega_e} \tau_1 (grad\, \tilde{\mathbf{N}})^T (grad\, \tilde{\mathbf{N}})\, d\Omega \\ \bar{\mathbf{f}}_{\mathbf{q}} &= \sum_e \int_{\Omega_e} \tau_1 (grad\, \tilde{\mathbf{N}})^T \cdot \mathbf{b}\, d\Omega \end{aligned}$$

We take $\tau_2 = 0$ and $\tau_1 = \frac{h^2}{12\nu}$. And we finally get an stable solution as we can see in Fig. 2.

1.3 Description of the problem (BC)

For the velocity, we are imposing Dirichlet conditions along $\delta\Omega = \delta([0,1]^2)$: $\mathbf{v} = (0,0)$ on $x = 0$, $x = 1$ and $y = 0$ and $\mathbf{v} = (1,0)$ on $y = 1$.

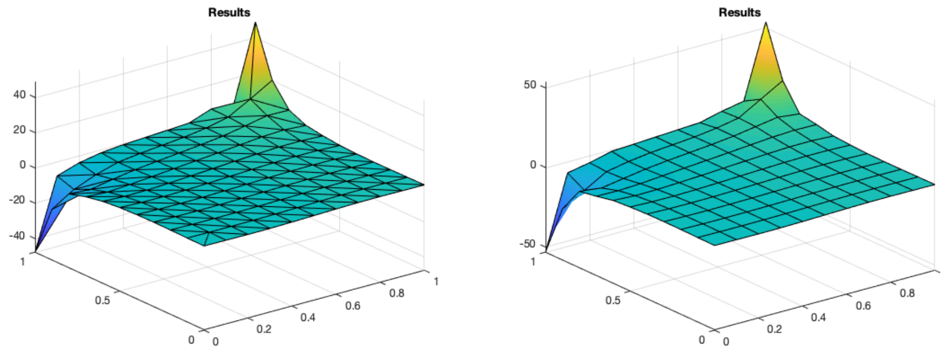

Figure 2: Resulting pressure after the stabilization method for the Q1Q1 elements (left) and P1P1 elements (right).

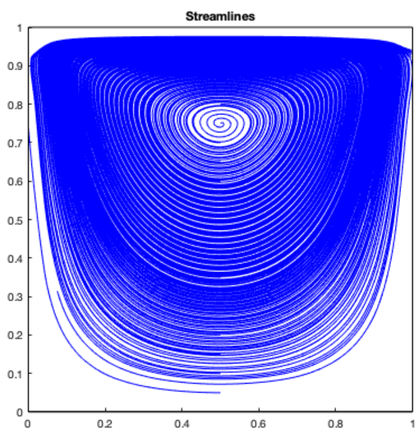

Figure 3: Streamlines. Integration of the movement of the particles of the fluid.

For the pressure, we are imposing natural boundary conditions ($\nabla p \cdot \mathbf{n} = 0$).

This is physically equivalent to a sink, since we are inducing a rotation along the confined fluid, that we can see represented in the ploted streamlines of Fig.3. This movement, produces and increase of the pressure on the top-right corner and decrease of it on the top-left corner as we can see in Fig.2.

2 Navier-Stokes problem

In this case, the **strong form** of the problem is:

$$\begin{aligned} -\nu\nabla^2\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= \mathbf{b} \text{ in } \Omega \\ \nabla\cdot\mathbf{v} &= 0 \text{ in } \Omega \end{aligned}$$

So we will have to add a convective term to the linear system 4, $\mathbf{C}(\mathbf{V}) = \int_\Omega (mat\,\mathbf{N})^T \mathbf{A}(\bar{\mathbf{V}})(grad\,\mathbf{N})\,d\Omega$, since

$$\begin{aligned} \omega(\mathbf{v}\cdot\nabla)\mathbf{v} &= (\omega_x\,\omega_y)\cdot\begin{pmatrix} \bar{v}_x\frac{\partial}{\partial x}v_x + \bar{v}_y\frac{\partial}{\partial y}v_x \\ \bar{v}_x\frac{\partial}{\partial x}v_y + \bar{v}_y\frac{\partial}{\partial y}v_y \end{pmatrix} && (6) \\ &\approx ((mat\,\mathbf{N})\cdot\mathbf{W})^T\cdot\begin{pmatrix} \bar{v}_x & 0 & \bar{v}_x & 0 \\ 0 & \bar{v}_y & 0 & \bar{v}_y \end{pmatrix}\cdot(grad\,\mathbf{N})\cdot\mathbf{V} && (7) \\ &\approx \mathbf{W}^T(mat\,\mathbf{N})^T\mathbf{A}(\bar{\mathbf{V}})(grad\,\mathbf{N})\mathbf{V} && (8) \end{aligned}$$

Thus we end up with the following non-linear system:

$$\begin{pmatrix} \mathbf{K}+\mathbf{C}(\mathbf{V}) & \mathbf{G}^T \\ \mathbf{G} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{V} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix} \tag{9}$$

To solve this non-linear system, we will use two iterative methods: Picard's method and Newton-Raphson's method.

2.1 Picard's method

In this case: we iterate $\Delta\mathbf{x}^{k+1} = \mathbf{F}^{-1}(\mathbf{x}^{k+1})(\mathcal{F} - F(\mathbf{x}^k)\mathbf{x}^k)$ until convergence on $\Delta\mathbf{x}^{k+1}$, where $\mathbf{x} = (\mathbf{V},\mathbf{P})$ and we are considering eq.9 as $F(\mathbf{x})\mathbf{x} = \mathcal{F}$. Where,

$$F(\mathbf{x}) = F(\mathbf{V},\mathbf{P}) = \begin{pmatrix} \mathbf{K}+\mathbf{C}(\mathbf{V}) & \mathbf{G}^T \\ \mathbf{G} & \mathbf{0} \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}$$

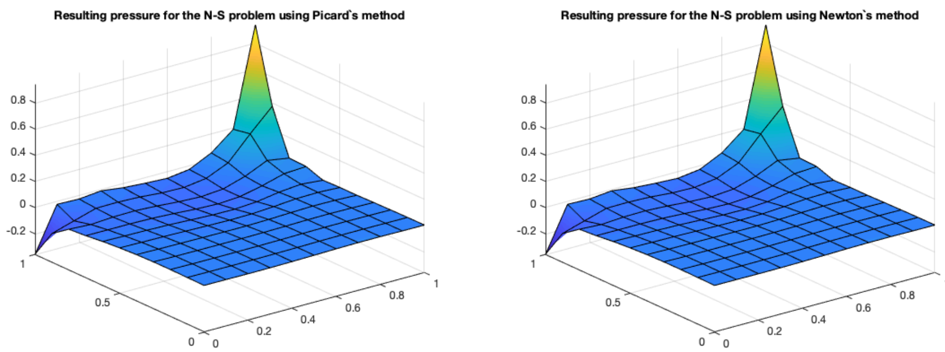

Figure 4: Pressure results for both methods for a 10 by 10 uniform mesh.

2.2 Newton's method

In this case: we iterate $\Delta\mathbf{x}^{k+1} = -\mathbf{J}(\mathbf{x}^k)(\mathbf{F}(\mathbf{x}^k)\mathbf{x}^k - \mathcal{F})$ until convergence on $\Delta\mathbf{x}^{k+1}$, where $\mathbf{J}(\mathbf{x}^k)$ is the Jacobian matrix of the system. To define $\mathbf{J}$ we have to compute $\frac{\partial \mathbf{C}(\mathbf{v})\mathbf{v}}{\partial \mathbf{V}}$. We know that

$$\begin{aligned}
\omega\mathbf{C}(\bar{\mathbf{v}})\mathbf{v} &= (\omega_x\ \omega_y)\cdot\begin{pmatrix} \bar{v}_x\frac{\partial}{\partial x}v_x + \bar{v}_y\frac{\partial}{\partial y}v_x \\ \bar{v}_x\frac{\partial}{\partial x}v_y + \bar{v}_y\frac{\partial}{\partial y}v_y \end{pmatrix} \\
&= (\omega_x\ \omega_y)\cdot\begin{pmatrix} \bar{v}_x & 0 & \bar{v}_y & 0 \\ 0 & \bar{v}_x & 0 & \bar{v}_y \end{pmatrix}\cdot\begin{pmatrix} \frac{\partial v_x}{\partial x} \\ \frac{\partial v_y}{\partial x} \\ \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial y} \end{pmatrix} = \mathbf{W}^T(mat\,\mathbf{N})^T\mathbf{A}(\bar{\mathbf{V}})(grad\,\mathbf{N})\mathbf{V} \\
&= (\omega_x\ \omega_y)\cdot\begin{pmatrix} \frac{\partial}{\partial x}v_x & \frac{\partial}{\partial y}v_x \\ \frac{\partial}{\partial x}v_y & \frac{\partial}{\partial y}v_y \end{pmatrix}\cdot\begin{pmatrix} \bar{v}_x \\ \bar{v}_y \end{pmatrix} = \mathbf{W}^T(mat\,\mathbf{N})^T\mathbf{B}(\mathbf{V})(mat\,\mathbf{N})\bar{\mathbf{V}}
\end{aligned}$$

Then $\frac{\partial \mathbf{C}(\bar{\mathbf{V}})\tilde{\mathbf{V}}}{\partial \mathbf{V}} = \frac{\partial \mathbf{C}(\bar{\mathbf{V}})\tilde{\mathbf{V}}}{\partial \tilde{\mathbf{V}}} + \frac{\partial \mathbf{C}(\bar{\mathbf{V}})\tilde{\mathbf{V}}}{\partial \bar{\mathbf{V}}} = (mat\,\mathbf{N})^T\mathbf{A}(\mathbf{V})(grad\,\mathbf{N}) + (mat\,\mathbf{N})^T\mathbf{B}(\mathbf{V})(mat\,\mathbf{N})$, so the Jacobian can be written as

$$J = \begin{pmatrix} \mathbf{K} + (mat\,\mathbf{N})^T\mathbf{A}(\mathbf{V})(grad\,\mathbf{N}) + (mat\,\mathbf{N})^T\mathbf{B}(\mathbf{V})(mat\,\mathbf{N}) & \mathbf{G}^T \\ \mathbf{G} & 0 \end{pmatrix} \tag{10}$$

As we can see in Fig. 4, both methods converge to the same solution. This solution has the same shape as the Stokes solution but it is smoother. Computing $|\Delta x|_\infty$ on the iterations for the two methods, we can see that they behave differently. As we can see in Fig. 5, Picard's method iterates 13 times and at each iteration the $log|\Delta x|_\infty$ decreases linearly. For the Newton's method, it only takes 5 iterations to get to the optimal solution since at each iteration the $log|\Delta x|_\infty$ decreases in a quadratic way.

This change in the way that the method converge makes also a big difference in the time it takes to obtain results. As we can see in table 2, for the Newton-Raphson's method it takes a lot less time than for the Picard's method.

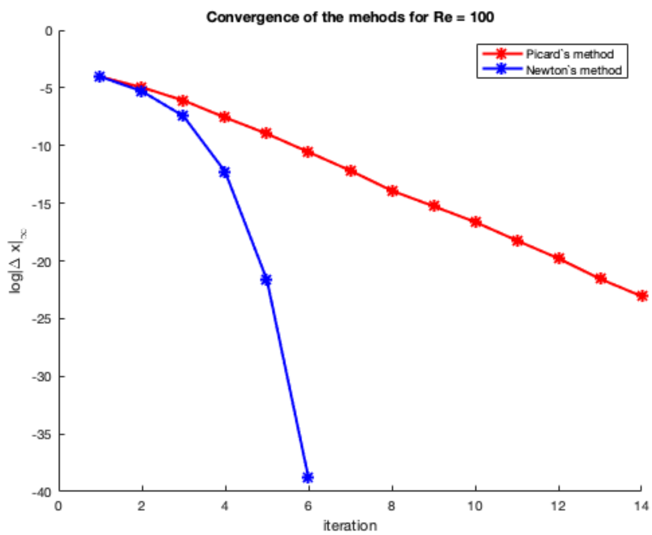

Figure 5: Relation between iterations and $log|\Delta x|_\infty$ of both methods for a 10 by 10 mesh ($Re = 100$).

Method	$n_{Elemets} = 5 \times 5$	$n_{Elemets} = 10 \times 10$	$n_{Elemets} = 20 \times 20$
Picard's	0.683686	1.386800	18.619847
Newton's	0.288000	1.005116	10.744418

Table 2: Time, in seconds, that it takes to solve the Navier-Stokes problem with the described $n_{Elements}$ for both methods