FEF Final Exam 2020

Master Computational Mechanics

Kiran Kolhe

Problem 1

$$-\nu \nabla^2 v + \nabla p = b \quad \text{in } \Omega$$
$$\nabla \cdot v = 0 \quad \text{in } \Omega$$
$$v = 0 \quad \text{on } \partial\Omega$$

ⓐ Yes, the proposed finite element pair is suitable to discretize the given equation.
The pair is Q_2Q_1 which has biquadratic velocity & bilinear pressure. The pressure & velocity interpolations are piecewise continuous.
Moreover, this pair satisfies the LBB condition which leads to a stable discretization. (The velocity approximation is greater than pressure approximation, here).

ⓑ Yes, the GLS stabilization is a suitable method for this problem.
This is implemented in order to ~~put~~ ~~some~~ ~~stabilization~~ ~~for the non-zero~~ have a non-zero term in the matrix equation to solve the stokes problem.
If this stabilization is not implemented, then we have a zero term on the diagonal because of the incompressibility condition.
So, using the stabilization makes the matrix stable by adding a certain non-zero term which produces desired stabilization for the pressure field.

(c)

Global size:

28 - nodes
8 - elements } $28 \times 8 = 224$ nodes → inside the elements.

Edges → $\underset{\text{nodes}}{7} \times 8$ edges $= 56$ nodes.

total size → $2(224) + 224 + 2(56)$

$= \underline{784 \text{ dof}}$

(d)

~~sizes,~~

~~28 nodes~~

e

$\boxed{1}$ Formulate the problem:

$$-\nabla\cdot(\nu\nabla v - p) = b \quad \text{in } \Omega$$
$$\nabla\cdot v = 0 \quad \text{in } \Omega$$
$$v = 0 \quad \text{on } \partial\Omega$$

$\boxed{2}$ Break the computational domain element by element.

$$-\nabla\cdot(\nu\nabla v - p) = b \quad \text{in } \Omega_e \quad (e = 1, \ldots n_{el})$$
$$\nabla\cdot v = 0 \quad \text{in } \Omega_e \quad (e = 1, \ldots n_{el})$$
$$v = 0 \quad \text{on } \partial\Omega$$

Add continuity of fluxes -

$$[\![v \otimes n]\!] = 0 \quad \text{on } \partial\Omega$$

$\boxed{3}$ Introduce a mixed variable to make the terms first order.

$$L + \sqrt{\nu}\,\nabla v = 0 \quad \text{in } \Omega_e$$
$$\nabla\cdot(\sqrt{\nu}L + p) = b \quad \text{in } \Omega_e$$
$$\nabla\cdot v = 0 \quad \text{in } \Omega_e$$
$$v = 0 \quad \text{on } \partial\Omega$$
$$[\![v \otimes n]\!] = 0 \quad \text{on } \partial\Omega$$

$\boxed{4}$ Solve the local problem & introduce the hybrid variable $(v_e = \hat{v})$,

$$\left.\begin{aligned} L_e + \sqrt{\nu}\,\nabla v_e &= 0 \\ \nabla\cdot(\sqrt{\nu}L_e + p_e) &= b \end{aligned}\right\} \Omega_e$$

$$\left.\begin{aligned} \nabla\cdot v_e &= 0 \\ v_e &= \hat{v} \end{aligned}\right\} \partial\Omega_e$$

Add the average of pressure on the boundaries

$$\frac{1}{\partial\Omega_e}\langle p_e, 1\rangle = \rho_e$$

5 Form the global problem

$$[\![v \otimes n]\!] = 0. \quad \text{on } \partial\Omega_e$$

But we already know the v is zero on boundaries.

6 Forming & solving weak forms of local & global problem would lead to formulation of matrix equations.

Solving these local & global equations will lead to the global problem

$$\begin{bmatrix} \hat{K} & \hat{G} \\ \hat{G}^T & 0 \end{bmatrix} \begin{Bmatrix} \hat{v} \\ \rho \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{\hat{v}} \\ \hat{f}_{\rho} \end{Bmatrix}$$

Problem 2

$$\frac{\partial u}{\partial t} - \nu \nabla^2 u + (u \cdot \nabla) u + \sigma u + \nabla p = f$$

$$\nabla \cdot u = 0$$

$$u = 0$$

$$u = u_0$$

(a) Time discritization,

$$\begin{cases} \dfrac{u^{n+1} - u^{n}}{\Delta t} - \theta \nu \nabla^2 u - (1-\theta) \nu \nabla^2 u + \theta (u^{n+1} \cdot \nabla) u^{n+1} + \\ (1-\theta) u^{n} \cdot \nabla u^{n} + \theta \nabla p^{n+1} + (1-\theta) \nabla p^{n} + \theta \sigma u^{n+1} \\ + (1-\theta) \sigma u^{n} = \theta f^{n+1} + (1-\theta) f^{n} \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

(b) Rearranging the above form,

$$\begin{cases} u^{n+1} - \Delta t\, \theta \left(\nu \nabla^2 u^{n+1} - (u^{n+1} \cdot \nabla) u^{n+1} + \sigma u^{n+1} \right) + \Delta t \nabla p^{n+1} \\ - \Delta t\, \theta f^{n+1} = u^{n} + \Delta t (1-\theta) \left(\nu \nabla^2 u^{n} - (u^{n} \cdot \nabla) u^{n} + \sigma u^{n} \right) \\ \qquad + \Delta t (1-\theta) f^{n} \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

For the weak form, considering test func. w & q.

$$\begin{cases} (w, u^{n+1}) - \Delta t\, \theta \left[a(w, u^{n+1}) - c(u^{n+1}; w, u^{n+1}) + (w, \sigma u^{n+1}) \right] \\ + \Delta t\, b(p^{n+1}, w) - \Delta t\, \theta (w, f) = (w, u^{n}) + \\ + \Delta t (1-\theta) \left[a(w, u^{n}) - c(u^{n}; w, u^{n}) + (w, \sigma u^{n}) \right] + \Delta t (1-\theta)(w, f^{n}) \\ b(q, u) = 0 \end{cases}$$

The terms in the previous equ. are calculated as $\rightarrow$ (sorry for the disregarding the order)

$C \rightarrow (u \cdot \nabla u) \rightarrow (w, (u \cdot \nabla) u) \rightarrow c(u; w, u) \qquad C(u)\, u$

$K \rightarrow \nu \nabla^2 u \rightarrow (w, \nu \nabla^2 u) = a(w, u)$

$$\rightarrow \int w\, \nu \nabla^2 u = \int \nabla \cdot w\, \nu \nabla u - \int \nabla w : \nu \nabla u$$

$$= \int_{\partial\Omega} w\, \nu\, \cancel{\nabla u \cdot n}^{\,0} - \int_{\Omega} \nabla w : \nu \nabla u \qquad K u$$

$R \rightarrow \sigma u \rightarrow (w, \sigma u) \qquad \sigma M u$

$\nabla \cdot u \rightarrow (q, \nabla \cdot u) \rightarrow b(q, u)$

$$\int_{\Omega} q \nabla \cdot u = \int_{\Omega} \nabla \cdot q u - \int_{\Omega} \nabla q \cdot u \quad \rightarrow \quad G u$$

$$= \int_{\partial\Omega} \cancel{q u \cdot n}^{\,0} - \int \nabla q\, u$$

Prs. grad $\rightarrow \nabla p \rightarrow (w, \nabla p) = b(p, w) \quad \rightarrow -G^T p$

Source term - $f \rightarrow (w, f)$

(c)

Now, using $w = N_i$, $q = \hat{N}_i$, $u = \Sigma N_i u_j$ $p = \Sigma \hat{N}_i p_i$

Putting $\theta = \frac{1}{2}$, in previous weak form & using above approximation & detinations given in previous step.

$$\left\{ \begin{array}{l} \left[m + \frac{\Delta t}{2}(K + C(u)^{n+1} - 6m)\right]u^{n+1} - \Delta t\, G^T p = \\ \qquad \left[m - \frac{\Delta t}{2}(K + C(u^n) - 6m)\right]u^n + \frac{\Delta t}{2} f^{n+1} + \frac{\Delta t}{2} f^n \\ -Gu = 0 \end{array} \right.$$

Matrix form becomes,

$$\begin{bmatrix} A & -\Delta t\, G^T \\ -G & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} A^n u^n + \frac{\Delta t}{2} f^{n+1} + \frac{\Delta t}{2} f^n \\ 0 \end{bmatrix}$$

where

$$A = m + \frac{\Delta t}{2}(K + C - 6m)$$

For BC $\rightarrow$ we know all boundaries are Dirichlet with $v_D = 0$.

(d) We may use either Picard method or Newton-Raphson Method.
Shown below is algorithm for Picard Method.

1 → Select initial guess x^0.

2 → Specify some tolerance for convergence

Now, compute approximation x^{k+1} till the convergence is achieved.

3 → Linear system of equation to be solved at each iteration,

$$A(x^k)\,x^{k+1} = b(x^k).$$

The reason I proposed Picard instead of Newton Raphson, is NR shows very good convergence only if the initial guess assumed is close enough to the solution.

(e) Yes, the methods are behaving as expected.
For low Re, for eg $Re = 100$, the newton Raphson method approaches convergence very fast as it has quadratic convergence.
Because, the important thing here for the NR method is that the initial guess chosen should be close enogugh to the solution.
As seen in the case of $Re = 1000$, the initial guess is far away from the solution, hence NR method does not behave well. & convergence of picard ~~In general,~~ can be seen much faster.
Also, picard has larger convergence radius than NR.

In general, to achieve good results a combination of both these methods also can be used. First starting with Picard & then when close to solution, then converging through NR method.