SHIVAPRASAD MARISWAMY FEF - Exam 25 July 2020

Exercise.1

convection - diffusion equation

$$-\nu u_{xx} + \beta u_x = 0 \quad , \quad x \in (0,1)$$
$$u = 0 \quad , \quad x = 0$$
$$u = 1 \quad , \quad x = 1 \qquad \text{with} \quad \nu = 0.05 \; , \; \beta = 2.$$

a) Let us call , $\Omega = (0,1)$.

In order to write the weak form weighting function 'w' is chosen such that $w=0$ at $x=0$ and $w=1$ at $x=1$.

Trial solution space, $S := \{ u \in H^1(\Omega) \,|\, u = 0 \text{ and } 1 \text{ at } x = 0 \text{ and } 1 \text{ respectively}\}$.

Test space, $\mathcal{V} := H^1_{\Gamma_D}(\Omega) = \{ w \in H^1(\Omega) \,|\, w = 0 \text{ and } 1 \text{ at } x = 0 \,\&\, 1 \text{ respt}\}$.

Using the divergence theorem on the diffusion term

$$\int_\Omega w(\beta u_x)\, d\Omega - \int_\Omega w_x(\nu u_x)\, d\Omega = wh\big|_{x=0} - wh\big|_{x=1}. \quad \text{for all } w \in \mathcal{V}$$

b) To write the discrete form of the Galerkin formulation, we can use the simpler version of representation of the weak form as

$$a(w^h, u^h) + c(w^h, u^h) = (w^h, h)\big|_{x=1} \qquad \text{for all } w^h \in \mathcal{V}^h.$$

Ω is discretised into Ω_e, $1 \le e \le n_{el}$. where, n_{el} = no. of element.

$$\text{Approximation}, \; u^h(x) = \sum_{A \in \eta \setminus \eta_D} N_A(x)\, u_A + \sum_{A \in \eta_D} N_A(x)\, u_D(x_A)$$

where N_A is the shape function associated with node number A and u_A is the nodal unknown value.

$$w^h \in \mathcal{V}^h := \operatorname*{span}_{A \in \eta / \eta_D} \{N_A\}.$$

Hence, the discrete weak form is

$$\sum_{B \in \eta \setminus \eta_D} [a(N_A, N_B) + c(N_A, N_B)] u_B = (N_A, \overset{\nearrow 0}{h})_{x=0} + (N_A, h)_{x=1}$$

$$- \sum_{B \in \eta_D} [a(N_A, N_B) + c(N_A, N_B)] u_D(x_B) \quad , \text{for all } A \in \eta \setminus \eta_D$$

With the imposition of u at $x=0$ and $x=1$, this system takes the below matrix form.

$$\boxed{(C + K)\, u = 0.}$$

$C = A^e c^e \qquad c^e_{ab} = \int_{\Omega_e} N_a (N'_{b_x})\, d\Omega$ → convection matrix.

$K = A^e K^e \qquad K^e_{ab} = \int_{\Omega_e} N'_{a_x} \cdot \nu N'_{b_x}\, d\Omega$ → diffusion matrix.

N'_a, N'_b → derivative.

i.e. $\frac{\partial N_a}{\partial x}, \frac{\partial N_b}{\partial x}$.

With the usage of shape functions,

$N_a(\xi) = \frac{1}{2}(1-\xi) \qquad N_b(\xi) = \frac{1}{2}(1+\xi)$

$$C^e = \frac{\beta}{2}\begin{pmatrix} -1 & +1 \\ -1 & +1 \end{pmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$K^e = \frac{\nu}{h}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} = \frac{0.05}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(c) If the domain is discretised with 12 uniform elements, there will be a better approximation for the diffusion term. But it doesnt affect the calculation of convective matrix. The overall results are expected to improve with the increase in element size. ~~Diffusion~~ The perturbation still sustains with this case. Needs stabilization.

(d) SUPG method is used to overcome the perturbation due to convective term dominance.

The new discrete form will be to to find $u^h \in S^h$ such that

~~$a(w^h, u^h) + c(w^h, u^h) + \sum_e \int_{\Omega^e} (w^h_x) \tau (u^h_x$~~

$$a(w^h, u^h) + c(a; w^h, u^h)$$

$$+ \sum_e \int_{\Omega^e} (a \cdot \nabla w^h) \tau [a \cdot \nabla u^h - \nabla \cdot (\nu \nabla u^h)] \, d\Omega = 0.$$

where, the stabilization parameter τ can be defined as

$$\tau = \frac{\bar{\nu}}{\|a\|^2}$$

with given $\bar{\nu} = \frac{\beta a h}{2}$ in 1D.

For the current problem,

$$a(w^h, u^h) + c(w^h, u^h) + \sum_e \int_{\Omega_e} (\nabla w^h) \tau [\nabla u^h - \nabla \cdot (0.05 \nabla u^h)] \, d\Omega = 0.$$

(e) The results with SUPG stabilization are free from perturbation seen before. The solution will not have oscillations caused by convective term.

Stabilization techniques are used to stabilize the convective term in a consistent manner.

Exercise-2

Stokes equation

$$-\nabla\cdot(\nu\nabla u - pI_{n_{sd}}) = b \text{ in } \Omega$$
$$\nabla\cdot u = 0 \text{ in } \Omega$$
$$u = u_D \text{ on } \Gamma_D$$
$$(\nu\nabla u - pI_{n_{sd}})n = g \text{ on } \Gamma_N.$$

a) The weak formulation is to find $(v,p) \in S\times Q$ such that

$$\nu(\nabla u, \nabla v) - (p, \nabla\cdot v) = \langle f, v\rangle \quad \forall\ v \in V$$
$$(q, \nabla\cdot u) = 0 \quad \forall\ q \in Q$$

with $V = H_0^1(\Omega)^d$ is the velocity space

$Q = L^2(\Omega)/R$ is the pressure space.

Including dirichlet and Neumann BCs.

$$a(u, \omega) + b(v,p) + b(v,q) = (\omega, b) + (\omega, t)_{\Gamma_N} \quad \forall\ (\omega,q) \in V\times Q$$

$$a(v,u) = \int_\Omega \nabla v : \nu\nabla u \, d\Omega.$$

ω, q are arbitrary weighting functions.

b) Higher order approximation of nominal functions for velocity, pressure and gradient of velocity results in the gain in accuracy for the same mesh sizes.

An optimal convergence of solution is expected with the P^2 (quadratic) polynomial approximations in HDG.

b) The interpolation of the pressure and velocity are in different orders in order to meet the LBB condition.

Dimension of velocity space $\mathcal{V}^h$ should be greater than the dimension of pressure space $\mathcal{Q}^h$.

Hence, the LBB condition may not be satisfied.

And the solution could be unstable.

d) Galerkin formulation of the given problem can be stated as

$$a(w^h, u^h) + b(v^h, p^h) = (v^h, b^h) + (w^h, t^h)_{\Gamma_N} - a(w^h, u_0^h)$$

$$b(u^h, q^h) = -b(u_0^h, q^h) \qquad \text{with } u^h \in \mathcal{V}^h$$

$$p^h \in \mathcal{Q}^h$$

Velocity approximation

$$u_i^h(x) = \sum_{A \in \eta / \eta_{D_i}} N_A(x)\, u_{iA}$$

$$u_{D_i}^h(x) = \sum_{A \in \eta_{D_i}} N_A(x)\, u_{D_i}(x_A)$$

where N_A, is the shape function at global node number A.

Pressure field interpolation,

$$p^h(x) = \sum_{\hat{A} \in \hat{\eta}} \hat{N}_{\hat{A}}(x)\, p_{\hat{A}}$$

where, $\hat{A}$ is the global pressure node number and $p_{\hat{A}}$ is the pressure value at $\hat{A}$ node.

Using GLS stabilization method

$$(-\nu\nabla^2 v + \nabla q,\ -\nu\nabla^2 u + \nabla p - b) = 0 \quad \forall\ (v,q) \in \mathcal{V}\times Q$$

is equivalent to

$$\begin{cases} (-\nu\nabla^2 v,\ -\nu\nabla^2 u + \nabla p - b) = 0 & \forall\ v \in \mathcal{V} \\ (\nabla q,\ -\nu\nabla^2 u + \nabla p - b) = 0 & \forall\ q \in Q \end{cases}$$

The stabilized discrete form;

find $u^h \in S^h$ and $p^h \in Q^h$, $\forall\ (v^h, q^h) \in \mathcal{V}^h \times Q^h$, such that

$$\begin{cases} a(v^h, u^h) + b(v^h, p^h) + \sum_{e=1}^{n_{el}} \tau_e(-\nu\nabla^2 v^h, -\nu\nabla^2 u^h + \nabla p^h - b^h)_{\Omega_e} \\ \qquad = (v^h, b) + (v^h, g^h)_{\Gamma_N} \\ b(u^h, q^h) - \sum_{e=1}^{n_{el}} \tau_e(\nabla q^h, -\nu\nabla^2 u^h + \nabla p^h - b^h)_{\Omega_e} = 0 \end{cases}$$

The stabilization parameter, $\tau_e = \alpha_0 \dfrac{h_e^2}{4\nu}$, h_e = element size

The discrete version of stokes problem assumes the following partitioned form

$$\begin{bmatrix} K+L & G \\ G^T + W & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ h \end{bmatrix}$$

K - viscosity matrix

L & W are from stabilization

G - discrete gradient operator.

G^T - discrete divergence operator

e) Yes, GLS stabilization is suitable for HDG discretization. ~~HD~~ Errors in HDG solution are highly sensitive to the magnitude of stabilization parameter.

f) Higher order approximation of for velocity than that of pressure is meaningful in the case of multiple approximation methods. GLS stabilization should be suitable for P^3 in velocity and P^2 in pressure polynomials.