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FEF FINAL 2020

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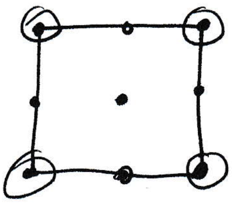
Problem 1

a) Given the stokes problem, are Q_2Q_1 elements suitable to discretize the problem?

Yes, Taylor Hood elements are known to be LBB stable, that is, they comply with the so called inf-sup condition



Q_2Q_1

○ p • $\bar{v}$

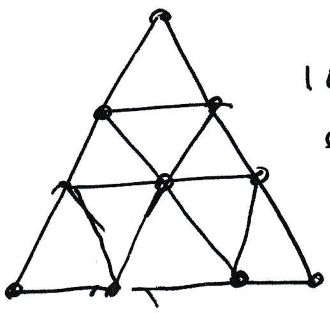
$$\inf_{\substack{q^h \in Q^h \\ q^h \neq 0}} \sup_{\substack{\bar{v}^h \in V^h \\ \bar{v}^h \neq 0}} \frac{b(\bar{v}^h, q^h)}{|\bar{v}^h|_1 \, |q|_0} \geq \beta > 0$$

for low Reynold numbers, meeting the LBB condition guarantees stability. Low Re is the basic assumption of Stoke's eqs. (inertia can be neglected).

b) If Q_2Q_1 elements are used, stabilization is unnecesary. If other elements (non-LBB) are used, GLS is appropriate as shown by Hughes & Franca (1987)

(2)

c) Each triangle has 28 nodes. There are 8 elements so assuming 2 velocity components (2D) we have 448 unknowns for velocity (nodes on the edges have different values in each element for DG.) To be LBB compliant, a necessary condition is that $\dim(Q^h) \leq \dim(V^h)$ (space for pressure should be less rich than the velocity space). To be safe, we could use a 10 noded element for pressure, then we would have $8 \times 10 = 80$ pressure values. Total unknowns would be 528 (including the nodes on the boundary).



10 noded element.

d) When implementing HDG we have the added complexity of the hybrid variable '$\check{u}$'

$n_{env} = 28$ (nodes per element for $\bar{v}$)

$n_{enp} = 10$ (" " " " p)

$n_{efv} = 18$ (nodes per on the face of each element for $\bar{v}$)

$n_{efp} = 10$ (ditto for p)

③ the hybrid variable is defined only on the faces of the elements, so for each element we have.

	# unknown	location
V_x	28	inside + face
V_y	28	inside + face
P	10	inside + face
$\check{u}_x$	18	face only
u_y	18	face only.

102 unknowns per element per time-step.

we included boundary node values

e) Sketch implementation of HDG.

① for each element write the unknowns in terms of the hybrid variable ($\check{u}$)

② Solve for $\check{u}$ imposing transmission condition $[\![\bar{m}\cdot\bar{q}]\!]=0$ between elements (naturally $[\![\bar{m}\cdot\bar{q}]\!]=-t$ on Γ_N)

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3 Solve the problem within each element, knowing the boundary conditions on its face. [great for parallelization]

4 Compute the postprocessed solution to obtain the primal variable.

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Problem 2

a) Write the time discrete problem using Crank-Nicholson:

Solution C.N. approximation for time uses $\theta = 1/2 \Rightarrow$

$$\frac{\bar{v}^{m+1} - \bar{v}^{m}}{\Delta t} = \frac{1}{2}\left[F^{m+1}(\dots) + F^{m}(\dots) \right]$$

in this case $F^{m} = \mu \nabla^2 \bar{v}^{m} - (\bar{v}^{m} \cdot \nabla)\bar{v}^{m} - \sigma \bar{v}^{m} - \nabla p^{m} + f$

However, as simple average of F^{m+1} and F^{m} contains non linear terms, a practical approach is used to evaluate some terms at different time steps. The most common approach is

$$\begin{cases} \dfrac{\bar{v}^{m+1} - \bar{v}^{m}}{\Delta t} + \underbrace{(\bar{v}^{m} \cdot \nabla) v^{m+1}}_{\text{semi-implicit}} - \mu \nabla^2 \bar{v}^{m+1/2} + \nabla p^{m+1/2} + \underbrace{\sigma \bar{v}^{m+1/2}}_{\text{porous resistance}} = f^{m+1/2} \\ \nabla \cdot \bar{v}^{m+1} = 0 \end{cases}$$

b) Find the weak form of 2.a)

looking now only at the steady part of the problem we have

$$-\mu \nabla^2 \bar{v} + (\bar{v} \cdot \nabla)\bar{v} + \nabla p + \sigma \bar{v} = f$$

multiply by test function w and integrate by parts

(6)

$$a(\bar{w}, \bar{v}) := \int_\Omega \nabla \bar{w} : \nu \nabla \bar{v} \, d\Omega$$ ← (eliminated the $\nabla^2 \bar{v}$ integrating by parts)

$$c(\bar{w}, \bar{v}, \bar{v}) := \int_\Omega \bar{w} \cdot (\bar{v} \cdot \nabla) \bar{v} \, d\Omega$$

$\Gamma_D = 0$, no traction conditions (Γ_N)

$$(\bar{w}, \bar{f}) = \int_\Omega \bar{w} \cdot \bar{f} \, d\Omega \qquad \underline{\underline{e(\bar{w}, p)}} = \int_\Omega p \, \nabla \cdot \bar{w} \, d\Omega$$

$$d(\bar{w}, \bar{v}) = \int_\Omega \bar{w} \cdot \sigma \bar{v} \, d\Omega$$ (porous medium term)

$$b(\bar{v}, q) = \int_\Omega q \, \nabla \cdot \bar{v} \, d\Omega$$ (continuity)

where the test functions used are q (scalar) and $\bar{w}$ (vector) such that:

$$\bar{w} \ni \{ \bar{v} \in H^1(\Omega) : \bar{v} = \bar{0} \text{ on } \Gamma_D \}$$ compact support H^1

$$q \ni \{ p \in L_2(\Omega) \}$$ square integrable in Ω

We have assumed there are no Neuman conditions.

the final system including the time derivative:

$$\begin{cases} \displaystyle\int_\Omega \bar{w} \cdot \left[\frac{\bar{v}^{n+1} - \bar{v}^{n}}{\Delta t} \right] d\Omega + a(\bar{w}, \bar{v}^{n+1/2}) + c(\bar{w}, \bar{v}^n, \bar{v}^{n+1}) + e(\bar{w}, p) \; \cdot \\ \qquad \underline{| \; d(\bar{w}, \bar{v}^{n+1}) = (\bar{w}, \bar{f}^{n+1})} \\ b(\bar{v}^{n+1}, q) = 0 \end{cases}$$

c) Discretize the two equations in 2b)

<u>solution</u>: the final system of equations will have the form:

$$\begin{bmatrix} \bar{\bar{A}} & \bar{\bar{G}}^T \\ \bar{\bar{G}} & 0 \end{bmatrix} \begin{bmatrix} \bar{v}_h^{m+1} \\ \bar{p}_h^{m+1} \end{bmatrix} = \begin{bmatrix} \bar{f}^{m+1} \\ \bar{0} \end{bmatrix}$$

with $\bar{\bar{A}} = \bar{\bar{M}} + \Delta t \left[\bar{\bar{K}} + C(\bar{v}_h^{m+1}) + \overbrace{D(\bar{v}^{m+1})}^{\text{POROUS MEDIUM TERM}}\right]$

where $\bar{\bar{K}} = \int_\Omega (\text{Grad}\,\bar{N})^T (\text{Grad}\,\bar{N})\, d\Omega$

$\bar{\bar{C}} = \int_\Omega (\text{grad}\,\bar{N})\, \bar{v}_h^m\, d\Omega$

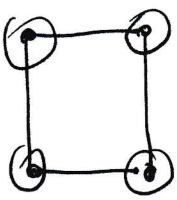
$\bar{\bar{D}} = \int_\Omega \sigma \bar{N}\, d\Omega$ ← new term.

and $\bar{\bar{G}} = -\int_\Omega \hat{N} D\, d\Omega$

$$\bar{v}_h = \sum_{j=1}^m \bar{v}_j N_j = \sum_{j=1}^m \begin{bmatrix} v_x^j \\ v_y^j \\ v_z^j \end{bmatrix} N_j \qquad N \ni V \;(H^1,\ N=0 \text{ on } \Gamma_D)$$

$$p_h = \sum_{j=1}^m p_j \hat{N}_j \qquad p \ni Q \;(L_2(\Omega))$$

Note Q_1Q_1 elements are <u>not</u> stable. if this space is used, the same form of stabilization (GLS, SGS, SUPG) will have to be added!



• p

• $\bar{v}$

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d) at each time step we have a non linear system of the type $\bar{\bar{K}}(\bar{x})\,\bar{x} = \bar{b}(\bar{x})$ where the coefficients of the matrix $\bar{\bar{K}}$ depend on the solution. A very robust and simple algorithm is <u>Picard's method</u>:

1. start with a guess $\bar{x}_0$
2. Calculate $\bar{\bar{K}}(\bar{x}_0), \bar{b}(\bar{x}_0) \Rightarrow$ obtain $\bar{x}_1$
3. update $\bar{\bar{K}}(\bar{x}_1), \bar{b}(\bar{x}_1) \Rightarrow$ obtain $\bar{x}_2$

$\vdots$

iterate untin $|\bar{x}_{n+1} - \bar{x}_n| <$ tolerance

e) Picards method converges in both cases although it is slower for the high Re case, as there is more motion of particles occurring for the same time step.

Newton shows quadratic convergence for low Re as expected, but fail to converge for high Re. This is because slope-based methods can fail miserably if the 1st derivative (in this case the Jacobian) is not 'pointing' in the direction of the solution

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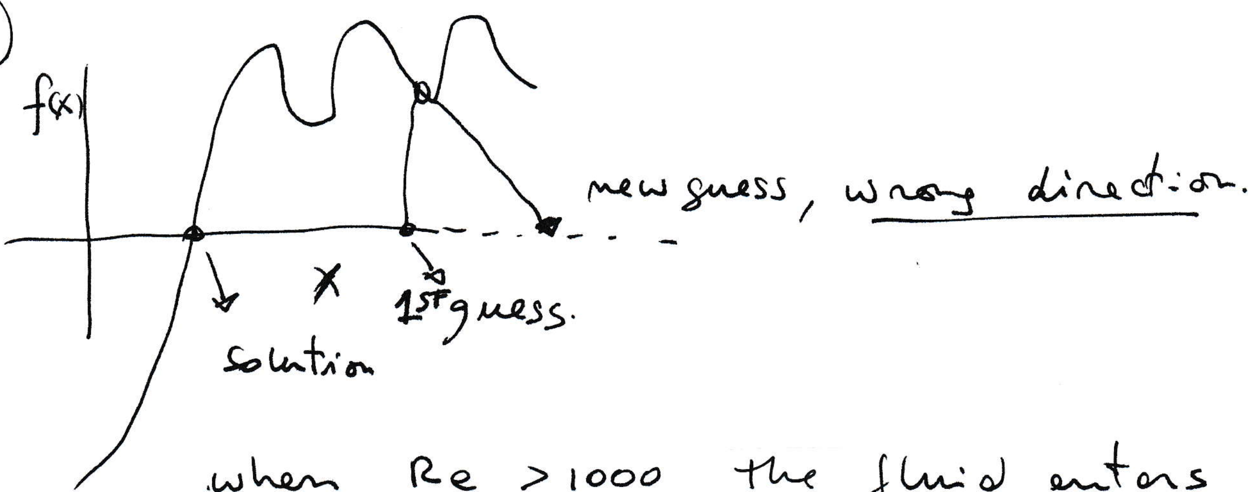

when $Re > 1000$ the fluid enters transition regime and becomes chaotic. it is reasonable that gradient methods should fail to find a solution. A combined approach (Picard first and Newton second) might be the best approach.